Quantum error correction, Operator Algebra, Representation Theory

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Joint work with Cordelia Li, Diane Pelejo, Sage Stanish.

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- In the context of quantum error correction, $E_1, \ldots, E_r$ are the error operators associated with the channel.

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$$\tilde{P} \to P_1P_2P_2 \to Q_1Q_2Q_3 \to (Q_1Q_2Q_3) \otimes (R_1R_2) \to ((Q_1Q_2Q_3), (M_1M_2)) \to (P_1P_2P_3) \to \tilde{P}.$$

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Ideally, $\quad \tilde{P} \to P_1 P_2 P_2 \to Q_1 Q_2 Q_3 \to P_1 P_2 P_3 \to \tilde{P}.$

Example: Bit-flip channel

Consider the bit-flip error $\tilde{\rho} \mapsto X\tilde{\rho}X^\dagger$ with $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ that will exchange the two classical states $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

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where $p_0, p_1, p_2, p_3 \geq 0$ summing up to 1,

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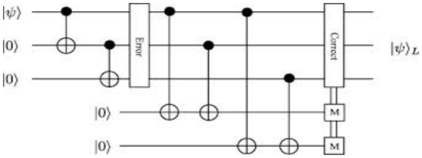

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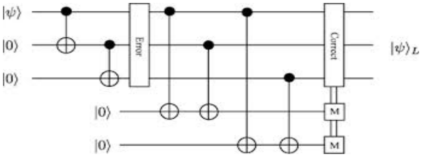

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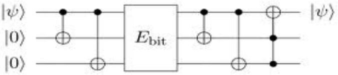

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- Let $\mathcal{A}$ be the algebra generated by $\{E_1, \ldots, E_r, E_1^\dagger, \ldots, E_r^\dagger\}$.
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$$U^\dagger \mathcal{A} U \subseteq (I_f \otimes M_g) \oplus M_{n-fg}.$$

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Thus $U^\dagger E_j U = \begin{pmatrix} I_{f_1} \otimes F_j & \\ & G_j \end{pmatrix}$ for $j = 1, \ldots, r$.
- If we encode $\tilde{\rho} \in M_f$ as $\rho = U \begin{pmatrix} \tilde{\rho} \otimes \sigma & \\ & 0_{n-fg} \end{pmatrix} U^\dagger$,

  then $\mathcal{E}(\rho) = U \begin{pmatrix} \tilde{\rho} \otimes \hat{\sigma} & \\ & 0_{n-fg} \end{pmatrix} U^\dagger$ so that $\tilde{\rho}$ is recoverable.

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- For example, we need to use the standard gates or basic gates available at the IBM online quantum computers: qiskit.

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- It is possible that only a finite number unitary W may occur, say, $W \in \{I_2, \sigma_x, \sigma_y, \sigma_z\}$, where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Frobenius formula

- One can use representation theory to decompose the algebra $\mathcal{A}$ generated by $\{W^{\otimes n} : W \in SU(2)\}$ as

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$n = 2$: $2^2 = 1 * 3 + 1 * 1$, no error correction.

$n = 3$: $2^3 = 1 * 4 + 2 * 2$, $(p, q, r) = (1, 1, 1)$.

$n = 4$: $2^4 = 1 * 5 + 3 * 3 + 2 * 1$, $(p, q, r) = (1, 1, 2)$ or $(1, 0, 3)$.

$n = 5$: $2^5 = 1 * 6 + 4 * 4 + 5 * 2$, $(p, q, r) = (2, 2, 1)$ or $(2, 1, 2)$.

For $n = 3$, the following two unitary matrices satisfy $U^\dagger \mathcal{A} U = I_2 \otimes M_2 \oplus M_4$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\ -\sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & \sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & \sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & \sqrt{\frac{1}{3}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ -\sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{3}} & 0 & 0 \\ -\sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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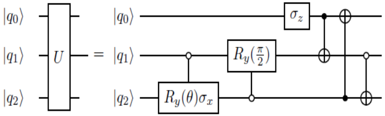
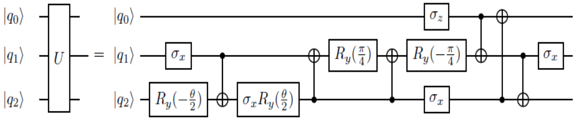
We are only able to decompose the second one into 5 standard gates $U_1, \ldots, U_5$ or 14 basic gates $V_1, \ldots, V_{14}$ with 6 CNOT gates.





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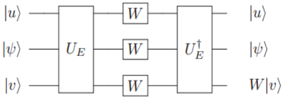

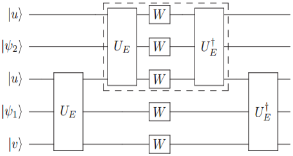

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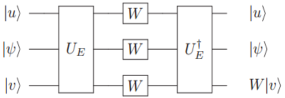

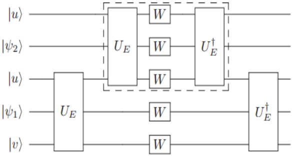

- We can extend the recursive scheme to protect k data qubits using 1 arbitrary states and k pure state.

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- The recursive scheme is useful before we can find a practical and efficient scheme.

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Thank you for your attention!