Conductance of a subdiffusive random weighted tree

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How fast does this sequence vanish?

Assume

- $u_n \searrow 0$;
- $u_n - u_{n+1} \leq$ [resp. $\geq$] $1/f(u_n)$;
- $f : (0, a) \to (0, \infty)$, $\searrow$, has an antiderivative $(-F)$.

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Use case 1

- $u_n - u_{n+1} \leq$ [resp. $\geq$] $C u_n^\alpha$ for some $\alpha > 1$, $0 < C < \infty$;
- $f(x) = C^{-1} x^{-\alpha}$;
- $F(x) = (C(\alpha - 1))^{-1} x^{-\alpha+1}$.

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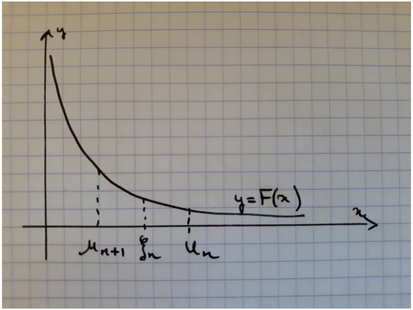
Use case 2

- $u_n - u_{n+1} \leq$ [resp. $\geq$] $Cu_n^2 \log 1/u_n$ for some $0 < C < \infty$;
- $f(x) = 1/(Cx^2 \log(1/x))$;
- $F(x) = ???$

Mean value theorem

$$u_n - u_{n+1} \leq \text{[resp. } \geq\text{]} \; 1/f(u_n).$$



$$\begin{aligned} F(u_{n+1}) - F(u_n) &= (u_n - u_{n+1}) f(\xi_n) \\ &\leq \text{[resp. } \geq\text{]} \; f(\xi_n)/f(u_n) \\ &\leq f(u_{n+1})/f(u_n) \\ &\geq 1 \end{aligned}$$

Cesàro's lemma

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$$\geq 1$$

$$\liminf \frac{1}{n} F(u_n) \geq 1$$

If $f(u_{n+1}) \sim f(u_n)$, then

$$\limsup \frac{1}{n} F(u_n) \leq 1$$

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$$\limsup \frac{1}{n} (C(\alpha-1))^{-1} u_n^{-\alpha+1} \leq 1$$

$$\liminf n^{1/(\alpha-1)} u_n \geq (C(\alpha-1))^{1/(\alpha-1)}$$

$$\limsup n^{1/(\alpha-1)} u_n \leq (C(\alpha-1))^{1/(\alpha-1)}$$

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- $u_n - u_{n+1} \leq$ [resp. $\geq$] $C u_n^2 \log 1/u_n$ for some $0 < C < \infty$;
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$$F(x) = -C^{-1} \int^x \frac{\mathrm{d}s}{s^2 \log(1/s)} = C^{-1} \int^{1/x} \frac{\mathrm{d}s}{\log s}$$

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However,

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{x}{\log x} = \frac{1}{\log x} - \frac{1}{(\log x)^2}.$$

So, $\mathrm{li}(y) \sim_{y \to \infty} y/\log y$ and $F(x) \sim_{x \to 0} C^{-1}/(x \log 1/x)$.

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(Some details later...)

$$\limsup (n \log n) u_n \leq C^{-1}$$

$$\liminf (n \log n) u_n \geq C^{-1}$$

Prologue

Conductance of a tree

Random walk on a tree

Galton-Watson trees

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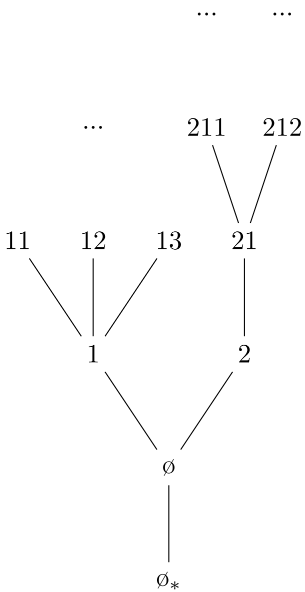
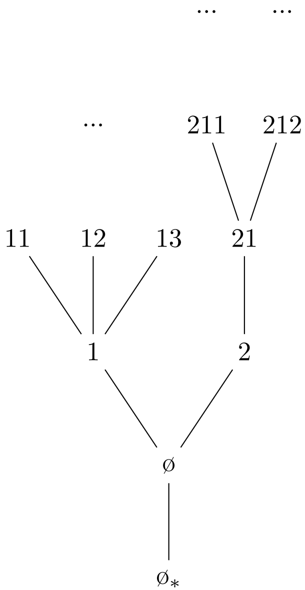
Planar trees, Neveu's formalism



- tree t: subset of the set of finite words on $\mathbb{N}^*$;
- rooted at $\varnothing$;
- artificial parent of the root: $\varnothing_*$;
- height in the tree: $|212| = 3$.
- parent: $(212)_* = 21$;
- number of children: $\nu_t(1) = 3$;
- t is infinite, without leaves (for simplicity) and locally finite: $\nu_t(x) \in [1, \infty)$, for all x in t.

Reindexed Subtree



t a tree and $x \in t$:

$$t[x] = \{y \in t : xy \in t\}.$$

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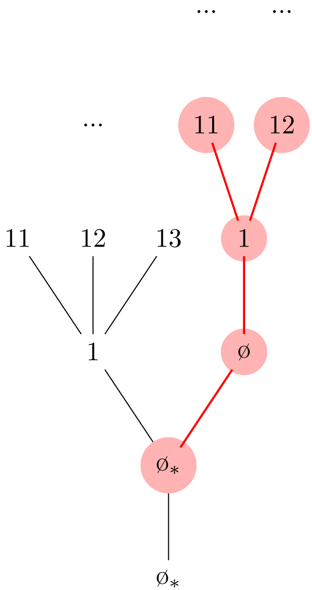

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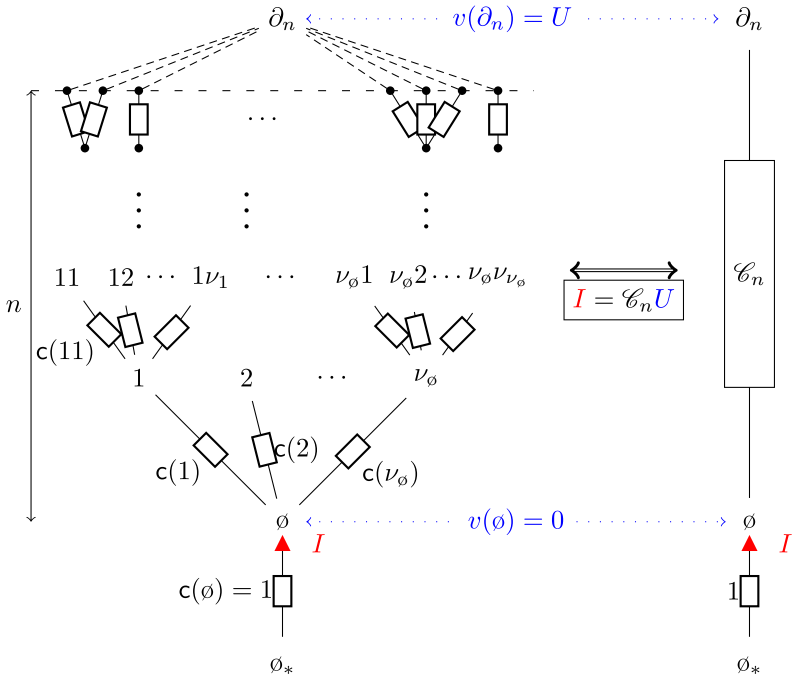

∂_n
$v(\partial_n) = U$
∂_n
n
11 12 $\cdots$ $1\nu_1$ $\cdots$ $\nu_\varnothing 1$ $\nu_\varnothing 2 \cdots \nu_\varnothing \nu_{\nu_\varnothing}$
$\mathscr{C}_n$
$I = \mathscr{C}_n U$
c(11)
1 2 $\cdots$ $\nu_\varnothing$
c(1)
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c($\nu_\varnothing$)
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I
I
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Electrical networks on trees

- Function $\mathsf{c} : t \setminus \{\varnothing_*\} \to (0, \infty)$;
- $\mathsf{c}(x)$: *conductance* of the edge $\{x_*, x\}$;
- ($\mathsf{c}(\varnothing)$ is always 1);
- $\mathsf{r}(x) = 1/\mathsf{c}(x)$: its *resistance*.

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- t_n: those vertices of t at height $\leq n$ ($n \in \mathbb{Z}_+$);
- For $x \in t_n$, $v(x) \in \mathbb{R}$: *potential* of the vertex x;
- For $\varnothing_* \neq x \in t_n$, $i(x) \in \mathbb{R}$: *current* flowing through the edge (x_*, x),
- Satisfying Ohm's law

$$v(x) - v(x_*) = \mathsf{r}(x) i(x) \quad \text{or, equivalently,}$$
$$i(x) = \mathsf{c}(x)(v(x) - v(x_*)).$$

Harmonicity

- Kirchhoff's current law at some vertex x:

$$i(x) = \sum_{j=1}^{\nu_x} i(xj);$$

- In term of potential (use Ohm's law):

$$\mathsf{c}(x)(v(x) - v(x_*)) = \sum_{j=1}^{\nu_x} \mathsf{c}(xj)(v(xj) - v(x))$$

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The potential at x is the c-weighted average of the potentials of its neighbors: v is *harmonic* at x.

Effective conductance

Fix the potential at some (sets of) vertices and harmonicity elsewhere:

$$v(x) = \begin{cases} 0 & \text{if } x = \emptyset; \\ U & \text{if } |x| = n; \\ \frac{1}{\pi(x)}\left(\mathsf{c}(x)v(x_*) + \sum_{j=1}^{\nu_x} \mathsf{c}(xj)v(xj)\right) & \text{if } 1 \le |x| \le n-1. \end{cases}$$

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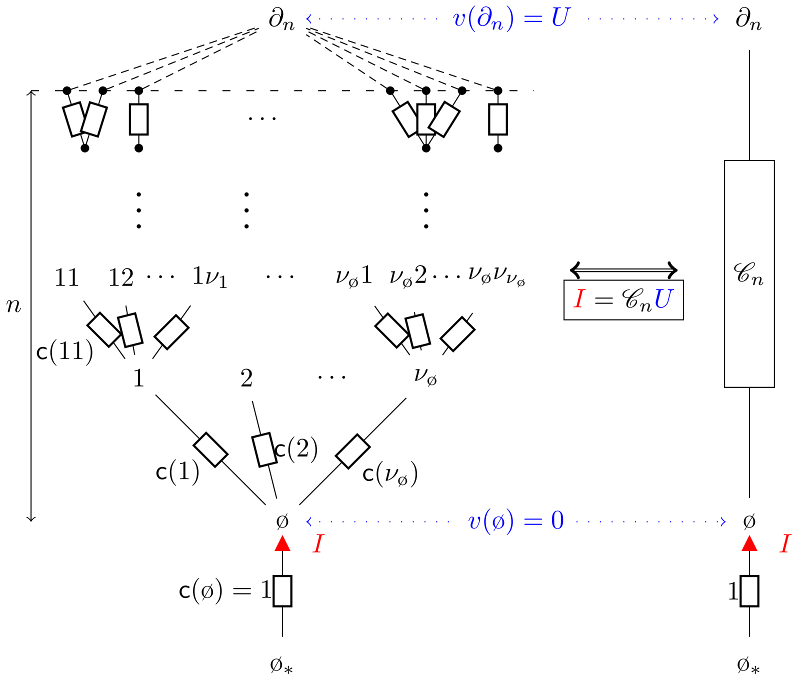
- Linear system, such a v exists and is unique;
- function $U \mapsto v$ clearly linear;
- $I = \sum_{j=1}^{\nu_\varnothing} i(j) = \sum_{j=1}^{\nu_\varnothing} \mathsf{c}(j)v(j)$, total current entering the network;
- $U \mapsto I$ is also linear;
- define $\mathscr{C}_n(t) = I/U$ (and $\mathscr{R}_n(t) = U/I$): *effective* conductance and resistance between $\varnothing$ and the n-th level of t.



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$\varnothing_*$
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Series and parallel laws

Conductance of an m-ary tree

Conductance of a m-ary tree (2)

$$\mathscr{C}_n(t) = \frac{m}{\lambda}\beta_{n-1}(t) = \frac{m}{\lambda}\frac{\mathscr{C}_{n-1}(t)}{1+\mathscr{C}_{n-1}(t)}.$$

In term of effective resistance:

$$\mathscr{R}_n(t) = \frac{\lambda}{m}(1+\mathscr{R}_{n-1}(t)) = \cdots$$

Define the effective resistance of the infinite tree t as

$$\mathscr{R}(t) = \lim_{n\to\infty}\mathscr{R}_n(t) \in (0,\infty] \quad \text{(it exists, see later).}$$

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Then $\mathscr{R}(t) < \infty \iff \lambda < m$.
Phase transition:

- $\lambda < m$: $\mathscr{R}_n(t) \to \mathscr{R}(t) < \infty$;
- $\lambda = m$: $\mathscr{R}_n(t) = n$, increases at linear speed;
- $\lambda > m$: $\mathscr{R}_n(t) \sim C(\lambda/m)^n$ increases at exponential speed.

Prologue

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Random walk and transition kernels

- $P : t \times t \to [0,1]$;
- $P(x,y)$ is the probability that a random walker located at the vertex x will go to y (in one step).
- for any $x \in t$, $\sum_{y \in t} P(x,y) = 1$;

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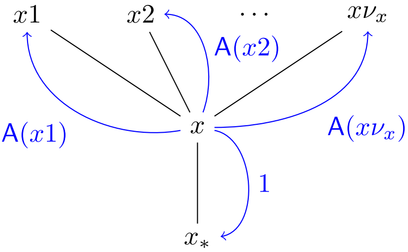
- $P : t \times t \to [0, 1]$;
- $P(x, y)$ is the probability that a random walker located at the vertex x will go to y (in one step).
- for any $x \in t$, $\sum_{y \in t} P(x, y) = 1$;
- nearest-neighbour random walk: $P(x, y) > 0$ if and only if y is a neighbour of x (its parent or one of its children);
- For $x \in t$, P_x: distribution of a path $(X_k)_{k \geq 0}$ in t, starting at x ($X_0 = x$) with transition kernel P.
- Example:

$$\begin{aligned} &P_\varnothing(X_1 = 1, X_2 = 12, X_3 = 1, X_4 = \varnothing) \\ &= P(\varnothing, 1)P(1, 12)P(12, 1)P(1, \varnothing). \end{aligned}$$

Random walk on a weighted tree



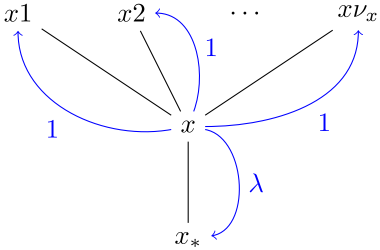
Transition kernel P:

$$P(x, x_*) = \frac{1}{1 + \sum_{j=1}^{\nu_x} \mathsf{A}(xj)};$$

$$P(x, xi) = \frac{\mathsf{A}(xi)}{1 + \sum_{j=1}^{\nu_x} \mathsf{A}(xj)} \quad i = 1, \ldots, \nu_x.$$

Particular case: λ-biased RW ($\mathsf{A} \equiv 1/\lambda$)

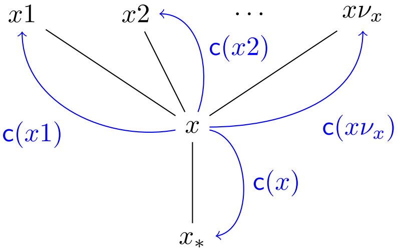
For some $\lambda > 0$,



$$P(x, x_*) = \frac{\lambda}{\lambda + \nu_x};$$
$$P(x, xi) = \frac{1}{\lambda + \nu_x} \quad i = 1, \ldots, \nu_x.$$

Simple RW if $\lambda = 1$.

Random walk on a tree with conductances



$$P(x, x_*) = \frac{\mathsf{c}(x)}{\pi(x)};$$

$$P(x, xi) = \frac{\mathsf{c}(xi)}{\pi(x)} \quad i = 1, \ldots, \nu_x.$$

One can always associate conductances to a RW on t:

- λ-biased random walk: $\mathsf{c}(x) = \lambda^{-|x|}$;
- in general, for weights A

$$\mathsf{c}(x) = \prod_{\varnothing \prec y \preceq x} \mathsf{A}(y).$$

Recurrence and transience

For $x \in t$, consider the times ($\min \emptyset = \infty$):

$$\tau_x = \min\{k \geq 0 : X_k = x\} \quad \text{and} \quad \tau_x^+ = \min\{k \geq 1 : X_k = x\}$$

Two regimes:

Recurrence Either for any x in t, $P_x(\tau_x^+ < \infty) = 1$: in this case, the walk almost surely visits every vertex infinitely many times;

Transience Or, for any x in t, $P_x(\tau_x^+ < \infty) < 1$: in this case, almost surely, the number of visits at any vertex is finite (and can be zero).

$\mathrm{RW} + \mathrm{EN} = \heartsuit$

Recall that there is a unique solution $v : t_n \to \mathbb{R}$ to

$$v(x) = \begin{cases} 0 & \text{if } x = \varnothing; \\ 1 & \text{if } |x| = n; \\ (P(x,x_*)v(x_*) + \sum_{j=1}^{\nu_x} P(x,xj)v(xj)) & \text{if } 1 \le |x| \le n-1. \end{cases}$$

Now fix some height $n > 0$ in t and let

$$f(x) = P_x(\tau^{(n)} < \tau_\varnothing).$$

Then, $f(\varnothing) = 0$ and $f(x) = 1$ if $|x| = n$.

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Then, $f(\varnothing) = 0$ and $f(x) = 1$ if $|x| = n$.
For $0 < |x| < n$, decompose with respect to the value of X_1:

$$\begin{aligned} f(x) &= P_x(\tau^{(n)} < \tau_\varnothing) \\ &= P_x(X_1 = x_*,\, \tau^{(n)} < \tau_\varnothing) + \sum_{j=1}^{\nu_x} P_x(X_1 = xj,\, \tau^{(n)} < \tau_\varnothing) \\ &= P(x, x_*)P_{x_*}(\tau^{(n)} < \tau_\varnothing) + \sum_{j=1}^{\nu_x} P(x, xj)P_{xj}(\tau^{(n)} < \tau_\varnothing). \end{aligned}$$

So $f(x) = v(x)$ for any $x \in t_n$.

$\mathrm{RW} + \mathrm{EN} = \heartsuit^2$

Finally, (recall that $U = 1$),

$$\begin{aligned}
\mathscr{C}_n(t) = I &= \sum_{j=1}^{\nu_\varnothing} \mathsf{c}(x) P_j(\tau^{(n)} < \tau_\varnothing) \\
&= \pi(\varnothing) \sum_{j=1}^{\nu_\varnothing} P(\varnothing, j) P_j(\tau^{(n)} < \tau_\varnothing) \\
&= \pi(\varnothing) P_\varnothing(\tau^{(n)} < \tau_\varnothing^+) \\
&\searrow_{n\to\infty} \pi(\varnothing) P_\varnothing(\tau_\varnothing^+ = \infty) = \mathscr{C}(t).
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Theorem

The random walk associated to the conductances $(\mathsf{c}(x))_{x\in t}$ is transient on t iff $\mathscr{C}(t) > 0$ (i.e. $\mathscr{R}(t) < \infty$).

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Corollary

λ-biased walk on an m-regular tree is transient iff $\lambda < m$.

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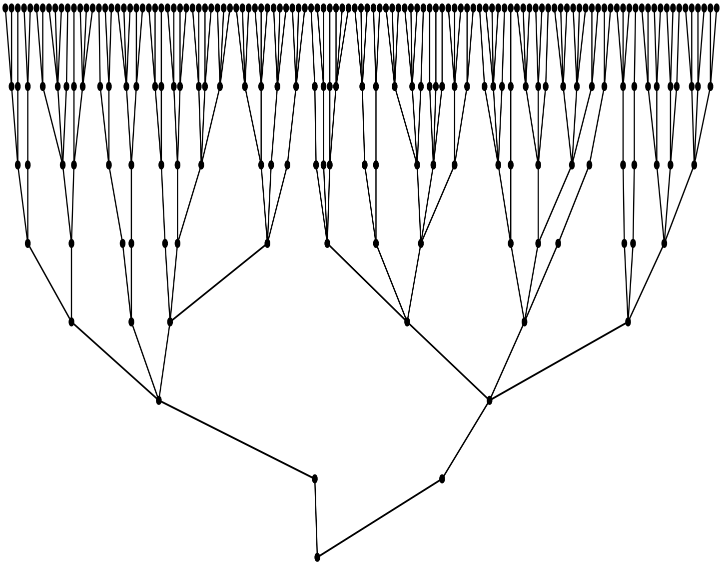

Galton-Watson trees (branching processes)

- Reproduction law: $(p_i)_{i\geq 0}$, non-negative numbers adding up to 1;
- $p_i = \mathbb{P}(\nu = i)$: probability that an individual has i children;
- assume $p_0 = 0$ (for clarity) and $p_1 < 1$ (for non-silliness);
- key-value: $\mathbb{E}[\nu] = \sum_{i\geq 1} ip_i =: m$, average number of children.

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- $p_i = \mathbb{P}(\nu = i)$: probability that an individual has i children;
- assume $p_0 = 0$ (for clarity) and $p_1 < 1$ (for non-silliness);
- key-value: $\mathbb{E}[\nu] = \sum_{i\geq 1} i p_i =: m$, average number of children.

We build an infinite random tree T:

- the root has $\nu_\varnothing \sim \nu$ children;
- each of its children then reproduces *independently* in the same way;
- and so on...

Branching property: "Subtrees above any height k are independent trees distributed as T and are independant of what happens below k."

How big is a GW tree?

For $n \geq 0$, let $Z_n(T)$ be the number of individuals at generation n in T. Then,

$$W_n(T) := \frac{Z_n(T)}{m^n} \to W(T).$$

Theorem (Kesten–Stigum, 1966)

$W(T) > 0$ *iff* $\mathbb{E}[\nu \log \nu] < \infty$.

λ-biased walk on a Galton-Watson tree

Let's try to do this...

In the same way as before (Series/Parallel laws),

$$\mathscr{C}_n(T) = \sum_{i=1}^{\nu_\varnothing} \lambda^{-1} \frac{\mathscr{C}_{n-1}(T[i])}{1+\mathscr{C}_{n-1}(T[i])}.$$

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By the branching property,

$$\begin{aligned}
\mathbb{E}[\mathscr{C}_n(T)] &= \mathbb{E}\Big[\sum_{i=1}^{\nu_\varnothing} \lambda^{-1}\Big] \mathbb{E}\Big[\frac{\mathscr{C}_{n-1}(T)}{1 + \mathscr{C}_{n-1}(T)}\Big] = \frac{m}{\lambda} \mathbb{E}\Big[\frac{\mathscr{C}_{n-1}(T)}{1 + \mathscr{C}_{n-1}(T)}\Big]. \\
&\leq \frac{m}{\lambda} \mathbb{E}[\mathscr{C}_{n-1}(T)] \\
&\to_{n\to\infty} 0 \quad \text{(exponentially fast)} \quad \text{if } \lambda > m.
\end{aligned}$$

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Fatou's lemma implies (in case $\lambda > m$),

$$\mathbb{E}[\mathscr{C}(T)] = \mathbb{E}[\lim \mathscr{C}_n(T)] \leq \liminf \mathbb{E}[\mathscr{C}_n(T)] = 0,$$

so $\mathscr{C}(T) = 0$, almost surely. Cases $\lambda = m$ and $\lambda > m$ are more difficult.

λ-biased walk on a Galton-Watson tree (2)

Theorem (Russell Lyons, 1990)

λ-biased RW on a GW-tree is almost surely transient iff $\lambda < m$.

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If $\mathbb{E}[\nu^2] < \infty$ and $\lambda = m$,

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- Perturbation around this critical case ($\mathsf{c}(x) = m^{-|x|} \times$ some r.v.) for an m-regular tree in Addario-Berry et al. and
- on a Galton-Watson tree (Chen–Hu–Lin 2018) with very precise asymptotics.

λ-biased walk on a Galton-Watson tree (3)

Open questions (as far as I know...):

- Does this result holds under the more reasonable assumption $\mathbb{E}[\nu \log \nu] < \infty$?
- What about the speed of the convergence $\mathscr{C}_n(T) \to \mathscr{C}(T)$ in the transient case?
- (Famous question:) In the transient case, does $\mathscr{C}(T)$ has a density (w.r.t. Lebesgue measure)?

Prologue

Conductance of a tree

Random walk on a tree

Galton-Watson trees

Conductance of a subdiffusive random weighted tree

Weighted Galton-Watson trees

Random finite sequence (whose length is also random):

$$\mathbf{A} = (A(1), A(2), \dots, A(\nu)) \in \bigsqcup_{k \geq 0} (0, \infty)^k.$$

$$\begin{array}{c} \varnothing \\ | \\ \varnothing_* \end{array}$$

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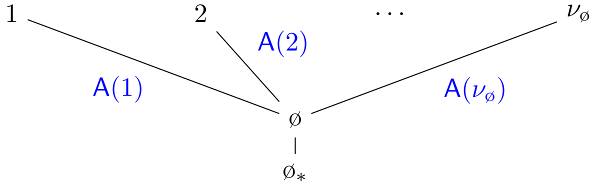

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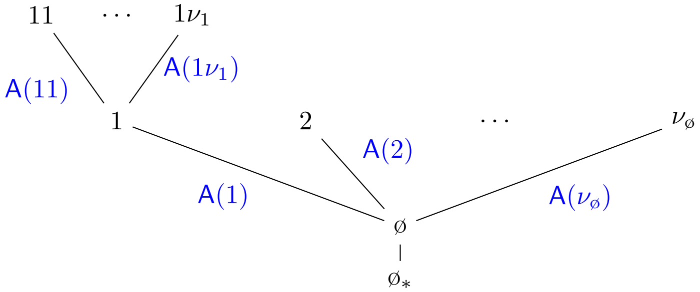

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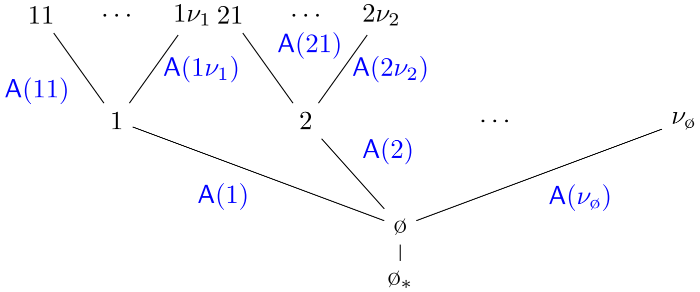

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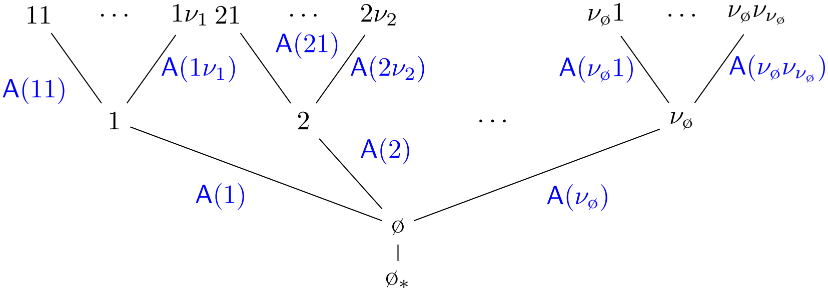

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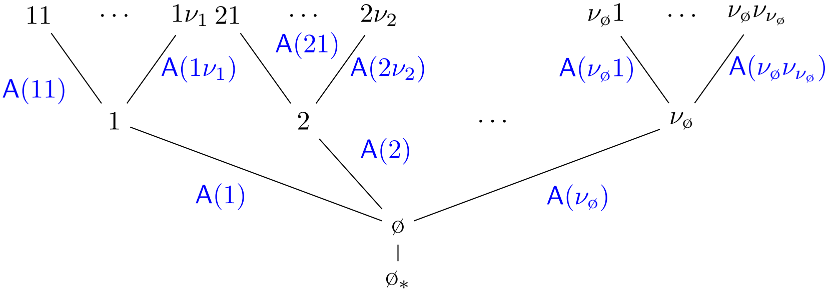

$\rightarrow T$, random tree with random weights.

Transience criterion

Theorem (Lyons–Pemantle 92, Faraud 11)

Consider, for $s \geq 0$,

$$\psi(s) = \log \mathbf{E}\Big[\sum_{i=1}^{\nu} A(i)^s\Big].$$

Then, the random walk on the weighted tree T *is transient iff*

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- Actually, many (4) "sub-regimes" in the recurrent case.
- Very rich model of random walk, popular in the RWRE community.

Normalized case

From now on, we assume

$$\psi(1) = \log \mathbb{E} \sum_{i=1}^{\nu} A(i) = 0. \qquad (H_{\text{norm}})$$

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$$\mathbb{E}[\mathscr{C}(T)] = \mathbb{E}\Big[\sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i)\Big] \mathbb{E}\Big[\frac{\mathscr{C}(T)}{1 + \mathscr{C}(T)}\Big] = \mathbb{E}\Big[\frac{\mathscr{C}(T)}{1 + \mathscr{C}(T)}\Big],$$

hence

$$\mathbb{E}\Big[\frac{\mathscr{C}(T)^2}{1 + \mathscr{C}(T)}\Big] = 0, \quad \text{thus} \quad \mathscr{C}(T) = 0 \quad \text{a.s.}$$

Biggin's theorem

Mandelbrot's martingale (1974):

$$M_n(T) = \sum_{|x|=n} \mathsf{c}(x) = \sum_{|x|=n} \prod_{\varnothing \prec y \preceq x} \mathsf{A}(x) \to M_\infty(T).$$

Theorem (Kahane–Peyrière 76, Biggins 77, Lyons 95)

Under the hypotheses

$$\psi'(1) \coloneqq \mathbb{E}\left[\sum_{i=1}^{\nu} A(i) \log A(i)\right] \in [-\infty, 0); \qquad (H_{\text{derivative}})$$

$$\mathbb{E}\left[\Big(\sum_{i=1}^{\nu} A(i)\Big) \log^+ \Big(\sum_{i=1}^{\nu} A(i)\Big)\right] < \infty, \qquad (H_{X \log X})$$

$M_\infty(T) > 0$ *a.s.*

Recursive distributional equation

$$\begin{aligned} M_{n+1}(T) &= \sum_{|x|=n+1} \prod_{\varnothing \prec y \preceq x} \mathsf{A}(x) \\ &= \sum_{i=1}^{\nu_\varnothing} \sum_{|x|=n+1,\, i \prec x} \mathsf{A}(i) \prod_{i \prec y \preceq x} \mathsf{A}(x) \\ &= \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i) M_n(T[i]). \end{aligned}$$

Therefore, M_∞ has the same distribution as

$$A(1)M_\infty^{(1)} + A(2)M_\infty^{(2)} + \cdots A(\nu)M_\infty^{(\nu)},$$

where $M_\infty^{(1)}$, $M_\infty^{(2)}$, ... are independent copies of M_∞, independent of the random vector $(A(1), \ldots, A(\nu))$.

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where $M_\infty^{(1)}$, $M_\infty^{(2)}$, …are independent copies of M_∞, independent of the random vector $(A(1), \ldots, A(\nu))$.
This characterizes the distribution of M_∞ among $\neq \delta_0$ probability measures (Quansheng Liu 97).

The "subdiffusive" regime

Recall that $\psi(1) = 0$ and $\psi'(1) < 0$. Consider

$$\kappa = \inf\{s > 1 : \psi(s) = 0\} \in (1, \infty].$$

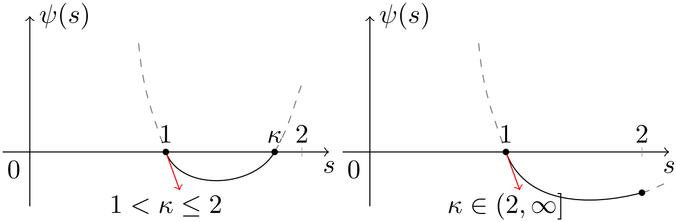

Figure: Schematic behavior of ψ under our hypotheses

$$\begin{aligned} &\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu} A(i)\Big)^{\kappa}\Big] + \mathbf{E}\Big[\sum_{i=1}^{\nu} A(i)^{\kappa} \log^{+} A(i)\Big] < \infty, \quad \text{if } 1 < \kappa \leq 2; \\ &\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu} A(i)\Big)^{2}\Big] < \infty, \quad \text{if } \kappa \in (2, \infty]. \end{aligned} \tag{H_κ}$$

Heuristics

$$\mathscr{C}_n(T) = \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i) \frac{\mathscr{C}_{n-1}(T[i])}{1 + \mathscr{C}_{n-1}(T[i])} \quad \text{and} \quad \mathscr{C}_n(T) \to 0.$$

We expect that $\mathbb{E}[\mathscr{C}_n] \to 0$ "not too fast" so
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$$\mathscr{C}_n(T)/u_n = \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i) \frac{\mathscr{C}_{n-1}(T[i])/u_n}{1+\mathscr{C}_{n-1}(T[i])} \approx \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i)\mathscr{C}_{n-1}(T[i])/u_{n-1},$$

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$$Y(T) = \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i) Y(T[i]),$$

so Y should be distributed as M_∞.

Our result

Theorem (R. 2020+)

Under the hypotheses (H_{norm}), $(H_{\mathrm{derivative}})$ *and* (H_κ), *as* n *goes to infinity,*

$$\mathscr{C}_n/\mathbb{E}[\mathscr{C}_n] \to M_\infty \quad a.s.$$

$$\mathbb{E}[\mathscr{C}_n] \asymp \frac{1}{n^{1/(\kappa-1)}} \quad \text{if } 1<\kappa<2;$$

$$\mathbb{E}[\mathscr{C}_n] \asymp \frac{1}{n\log n} \quad \text{if } \kappa=2 \text{ and}$$

$$\mathbb{E}[\mathscr{C}_n] \sim \frac{1}{n\mathbb{E}[M_\infty^2]} \quad \text{if } \kappa>2.$$

Lower bound

$$\mathscr{C}_n(T) = \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i) \frac{\mathscr{C}_{n-1}(T[i])}{1 + \mathscr{C}_{n-1}(T[i])}$$

so (branching property)

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Actually, $\leq$ is rigorous (some very magic inequality involved).

Lower bound (2)

Some knowledge of M_∞ (tail probability estimates) allow to compute, as a goes to infinity,

$$\mathbb{E}\Big[\frac{M_\infty^2}{1/a + M_\infty}\Big] \begin{cases} \asymp a^{1-\kappa} & \text{if } 1 < \kappa < 2; \\ \asymp a^{-1}\log a & \text{if } \kappa = 2; \\ \sim \mathbb{E}[M_\infty^2]/a & \text{if } \kappa > 2. \end{cases}$$

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and we can use our mean-value theorem based inequalities to conclude.

Thank you slide

Thank you for your attention!

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