COVERING NUMBERS OF GROUPS

(joint w/ Martino Garonzi & Luise-Charlotte Kappe)

Let G be a group and

$\mathcal{H}$: some collection of proper subgroups of G

If $G = \bigcup_{H \in \mathcal{H}} H$, then $\mathcal{H}$ is a cover of G

A cover of size n (containing n subgroups) is minimal if no cover of G contains fewer than n subgroups

DEF (Cohn 1994) $\sigma(G)$ is the size of a minimal cover (supposing one exists!)

(BAD) EX If G is cyclic, then G has no cover!

No proper subgroup contains a generator.

On the other hand, if G is <u>not</u> cyclic,

then $$G = \bigcup_{g \in G} \langle g \rangle$$

(In particular, $\sigma(G)$ is well defined & finite if G is finite & noncyclic)

"NATURAL" QUESTIONS

① Given a (finite) group G, what is $\sigma(G)$?

② For which integers $n \in \mathbb{N}$ does there exist a finite group G w/ $\sigma(G) = n$?

Work toward ① has been done for many "families" of groups, although *many* cases are still open!

This talk will focus on ②.

EXERCISE: Prove that no group G is the union of exactly two proper subgroups.

<u>EX</u> What $\sigma(G) = 3$?

Let $G = C_2 \times C_2 = \{(0,0), (1,0), (0,1), (1,1)\}$

operation: addition modulo 2 coordinatewise

<u>Subgroup Lattice</u>:

$$\begin{tikzcd}[column sep=small]
& C_2 \times C_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle (1,0) \rangle \arrow[dr, dash] & \langle (0,1) \rangle \arrow[d, dash] & \langle (1,1) \rangle \arrow[dl, dash] \\
& \langle (0,0) \rangle &
\end{tikzcd}$$

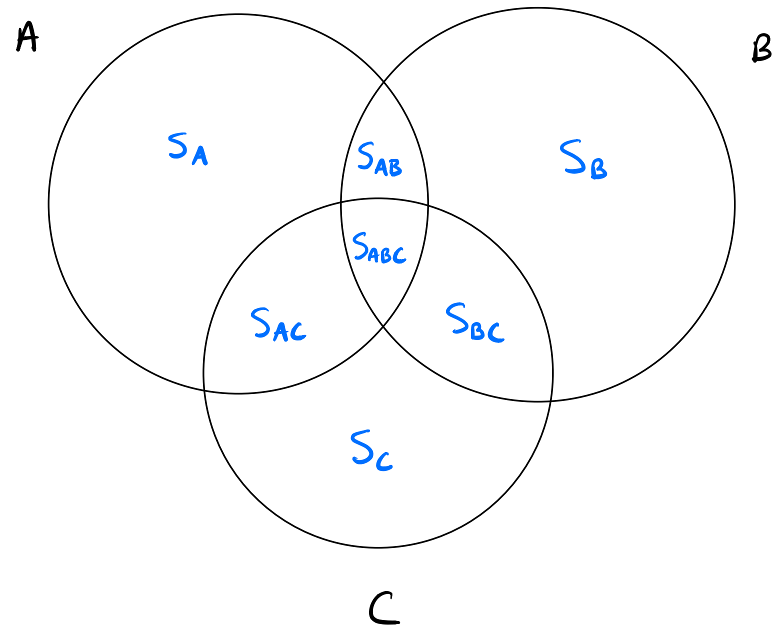
so: $\boxed{\sigma(C_2 \times C_2) = 3}$

Can we characterize groups w/ $\sigma(G) = 3$?

(Scorza, 1926)

Suppose $\sigma(G) = 3$, i.e., $G = A \cup B \cup C$

for subgroups A, B, C.



Suppose $x \in S_{BC}$ (so: $x \in B \cap C$, not in A)

Take $a \in S_A$. G grp: $ax \in G$

If $ax \in B \Rightarrow ax = b,\ b \in B$

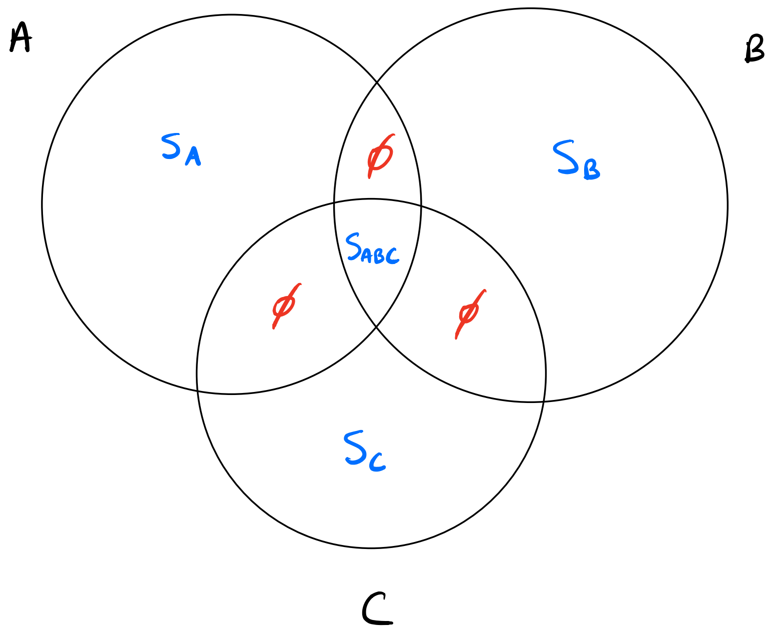
$\Rightarrow a = bx^{-1} \in B \Rightarrow \Leftarrow$ so $ax \notin B$

Similarly, $ax \notin C$

So: $ax \in A \Rightarrow x \in A \Rightarrow \Leftarrow$

So: $S_{BC} = \emptyset$

(Similarly, $S_{AB} = \emptyset$, $S_{AC} = \emptyset$)



If $a \in S_A$ and $b \in S_B$, then $ab \notin A \cup B$

$$\Rightarrow \quad ab \in S_C$$

so: $S_A S_B \subseteq S_C$ $\quad$ (and $S_A S_C \subseteq S_B$, $S_B S_C \subseteq S_A$)

Conversely, if $c \in S_C$, then, for any $b \in S_B$,

$$a = cb^{-1} \in S_C S_B \subseteq S_A$$

$$\Rightarrow \quad c = ab \in S_A S_B$$

$$\Rightarrow \quad S_C = S_A S_B \quad (= S_B S_A)$$

Similarly, $S_B = S_C S_A = S_A S_C$,

$$S_A = S_B S_C = S_C S_B$$

Since $S_A = S_B S_C$,

$$S_A^2 \subseteq S_A(S_B S_C)$$
$$= (S_A S_B) S_C \subseteq S_C^2, \text{ etc.}$$

so: $S_A^2 = S_B^2 = S_C^2 = S_A S_B S_C$
$$= S_{ABC}$$

For any $a \in S_A$,

$$a \cdot S_{ABC} = S_{ABC} \cdot a = S_A$$
$$b \cdot S_{ABC} = S_{ABC} \cdot b = S_B$$
$$c \cdot S_{ABC} = S_{ABC} \cdot c = S_C$$

<u>so</u>: $S_{ABC} \trianglelefteq G$

<u>and</u> S_A, S_B, S_C, S_{ABC} are the four distinct cosets of S_{ABC} in G

so: $G/S_{ABC} \cong C_2 \times C_2$

On the other hand, if we take a natural homomorphism $\phi: G \to G/N$
$(N \lhd G)$

and $\overline{H_1}, \ldots, \overline{H_k}$ is a cover of G/N,

then $\phi^{-1}(\overline{H_1}), \ldots, \phi^{-1}(\overline{H_k})$ is a cover of G!

$\sigma(G) \leq \sigma(G/N)$!

<u>THM</u> (Scorza 1926) $\sigma(G) = 3$ iff there is a surjective homomorphism
$\phi: G \to C_2 \times C_2$.

<u>THM</u> (B.H. Neumann 1954)
A group is the union of finitely many proper subgroups <u>iff</u> it has a finite, noncyclic homomorphic image.

<u>THM</u> (1) (Cohn 1994) For every prime p and integer $d \in \mathbb{N}$, there exists a finite group G w/ $\sigma(G) = p^d + 1$.

(EX: $\sigma(AGL(1,p^d)) = p^d+1$)

(2) (Tomkinson 1997) If G is a solvable group, $\sigma(G) = \underbrace{p^d+1}_{\text{exactly which } p,d \text{ depend on "chief series" of } G}$

so: $\cancel{2}$

3	4	5	6	7
2^1+1	3^1+1	2^2+1	5^1+1	??
✓	✓	✓	✓	
$C_2 \times C_2$	S_3	A_4	D_5	
			$\lvert D_5\rvert = 10$	

THM (Tomkinson 1997) There is no group G w/ $\sigma(G) = 7$.

Going further:

8	9	10	11
7^1+1	2^3+1	3^2+1	??
✓	✓	✓	

THM (Detomi, Lucchini 2008) There is no group G w/ $\sigma(G) = 11$.

A little further:

	12	13	14	15
	11^1+1	??	13^1+1	??
	✓		✓	

(Abdollahi, Ashrafi, Shaker 2007) $\sigma(S_6) = 13$

(Bryce, Fedri, Serena 1999) $\sigma(PSL(3,2)) = 15$

$\cong GL(3,2)$

What's currently known?

<u>THM</u> (Garonzi 2013) The integers between 16 and 25 that are <u>not</u> covering numbers are 19, 21, 22, 25.

<u>New Results</u>

<u>THM</u> (Garonzi, Kappe, S. 2020+) [Determined all integers between 26 and 129 that are <u>not</u> covering numbers]

(57 such integers)

<u>CONJ</u> There are infinitely many integers that are <u>not</u> covering numbers; further,

we conjecture

$$\lim_{n\to\infty} \frac{\{k \le n : \sigma(G) = k \text{ for some group } G\}}{n} = 0$$

THM (GKS 2020+) Let $q = p^d$ be a prime power and $n \ge 2$ ($n \ne 3$) be an integer. Then, there exists a group G such

$$\sigma(G) = \frac{q^n - 1}{q - 1}$$

$$\left(\sigma(AGL(n-1, q)) = \frac{q^n-1}{q-1}, \quad n \ge 2, \quad n \ne 3 \right)$$

Q: How do you determine which integers are covering numbers?

STRATEGY: Determine σ-elementary groups, that is, we determine the groups G such that $\sigma(G) < \sigma(G/N)$ for every nontrivial normal subgroup N.

THM (GKS 2020+) Let G be a nonabelian σ-elementary group w/ $\sigma(G) \leq 129$.

Then, G is primitive and monolithic

G has a unique minimal normal subgroup

w/ degree of primitivity ≤ 129, and the smallest degree of primitivity is $\leq \sigma(G)$

G is primitive of degree n if G is transitive on Ω, $|\Omega| = n$, and G preserves no nontrivial partition of Ω.

So, "all" that remains is calculating (or @ least bounding) $\sigma(G)$ for all such primitive & monolithic groups w/ degree of primitivity ≤ 129

While there are a few overlaps,

GAP: 1241 such groups

Techniques:

- Linear programming
  (great if $|G| < 500000$)

- Greedy algorithm that has a fast verification method
  $\underline{\underline{\&}}$ bounds $\sigma(G)$

<u>EX</u> Using this algorithm,

$$\sigma(A_{11} \text{ w/ } S_2) = 6\,380\,772$$