4 short talks (2 by me and 2 by Shintaro Nishikawa) on C^*-algebras and $L^2(G/H)$. They'll feature at least as many questions as results ... I hope that Sococo will be a good venue to discuss those questions...

This is joint work (to the extent that any work has been done, so far) with Alexander Afgoustidis, Peter Hochs, and Shintaro.

Some remedial C^*-algebra theory...

The set $\hat{A}$ of irreducible reps (up to equivalence) of a C^*-algebra A is a topological space with open sets

$$\hat{J} \subseteq \hat{A} \qquad (J \text{ a } C^*\text{-algebra ideal in } A)$$

$$\hat{J} = \{\pi \in \hat{A} \mid \pi[J] \neq 0\}$$

The complementary closed sets are

$$\widehat{A/J} \subseteq \hat{A}$$

$$\widehat{A/J} = \{\pi \in \hat{A} \mid \pi[J] = 0\}$$

(There are other ways of describing the topology, and we'll encounter one of them soon.)

The unitary dual (of a reductive group) and the tempered dual

There is a C*-algebra completion $C^*(G)$ of $L^1(G)$ with $\widehat{C^*(G)} = \widehat{G}$.

<u>Theorem</u> (Harish-Chandra, Cowling-Haagerup-Howe) The tempered dual of G is a closed subset of $\widehat{G}$.

So there is a quotient of $C^*(G)$ whose irreducible representations are precisely the tempered irreducible reps. of G. According to C, H & H it is

$$\boxed{C^*_r(G) = \text{Image}\left(C^*(G) \longrightarrow \mathcal{B}(L^2(G))\right)}$$

A (probably tiresome) reminder: the K-theory of $C^*_r(G)$ is considered to be interesting (by me and others) and there is a simple, beautiful formula for it (Connes & Kasparov)

But (another reminder) $L^2(G)$ is supposed to be viewed (by right-thinking people) as a representation of $G \times G$, not of G...

$$L^2(G) = \int_{\widehat{G}} H_\pi \otimes H_\pi^* \, d\mu(\pi)$$

$$L^2(G) = L^2((G \times G)/\Delta G)$$

- Can one construct a C*-algebra taking this point of view?
- If so, what about its K-theory? Is it the right K-theory?
- What about $L^2(G/H)$ in (more) generality?

Symmetric space

Group C^*-algebras, other convolution algebras, other options

Should we really be looking for a C^*-algebra for $L^2(G/H)$ (or even just $L^2((G\times G)/\Delta G)$)?

We're not sure.

We would like to use C^*-algebras to examine the topology — and even the K-theory — of the tempered dual of G/H. But...

- There are complicated phenomena not present in the "group case" of $(G\times G)/\Delta G$ that perhaps defy a simple topological description.

- In the group case there are various perspectives on the Connes–Kasparov isomorphism:
    - Dirac operators
    - Mackey deformation
    - Pseudodifferential operators
    - Casselman's Schwartz algebras
    - . . .

  and it is not clear which, if any, of these should be pursued in the general G/H case.

Image of the biregular representation

The image of the morphism

$$C^*(G\times G) \longrightarrow B(L^2(G))$$

is a quotient of $C^*(G\times G)$ and so it corresponds to a closed subset of $\widehat{G\times G} \cong \widehat{G}\times\widehat{G}$ (the latter homeomorphism is a little theorem of Wulfsohn).

What is this closed subset?

It is not $\{\pi\otimes\pi^* : \pi\in\widehat{G}_r\}$, in general.

tempered dual of G

For instance, for $SL(2,\mathbb{R})$ this closed set contains all four tensor products

$$\pi_+\otimes\pi_+^*,\ \pi_-\otimes\pi_+^*,\ \pi_+\otimes\pi_-^*,\ \pi_-\otimes\pi_-^*$$

obtainable from the limits of discrete series.

All four are in the closure of the tensor products of odd principal series $\pi_\nu\otimes\pi_\nu^*$, as one can see using:

Lemma A net of unirreps $\{\pi_\alpha\}$ of a group converges to π if and only if there are matrix coefficient functions φ_α for π_α and φ for π with $\varphi_\alpha \longrightarrow \varphi$ uniformly on compact subsets of the group.

This is a problem (probably). The two "extra" representations create two extra generators in K-theory.

Topology of the twisted diagonal in $\widehat{G}\times\widehat{G}$

The twisted diagonal is

$$\{\pi\otimes\pi^* \mid \pi\in\widehat{G}_r\}\subseteq\widehat{G}\times\widehat{G}$$

Theorem If G is linear real reductive, then the twisted diagonal is a locally closed subset of $\widehat{G}\times\widehat{G}$ (intersection of a closed subset and an open subset).

Before explaining a proof (which may or may not be a good proof), here is the important consequence for us.

Consider

$$F\cap U\subseteq\widehat{A}$$

closed set — open set — dual of some C^*-algebra

- The closed set corresponds to a quotient $A\to B$
- The open set corresponds to an ideal $J\to A$
- The image of J in B — a subquotient of A — is a C^*-algebra with spectrum $F\cap U$.

Proof of the Theorem The proof uses the maximal compact subgroup $K \subseteq G$, and relies on the fact that if $\sigma \in \widehat{K}$ then the set

$$\{ \pi \in \widehat{G} \mid \pi|_K \text{ includes } \sigma \}$$

is an open subset of $\widehat{G}$. It corresponds to the ideal

$$J = C^*(G)\, p_\sigma\, C^*(G) \subseteq C^*(G).$$

σ-isotypical projection

This will be applied to $K \times K \subseteq G \times G$.

The following set is closed:

$$\left\{ \pi_1 \otimes \pi_2^* \;\middle|\; \begin{array}{l} \pi_1 \,\&\, \pi_2 \text{ are summands of the same} \\ \text{cuspidal unitary principal series rep}^{\underline{n}} \end{array} \right\}$$

We want to exclude those $\pi_1 \otimes \pi_2^*$ for which $\pi_1 \not\cong \pi_2$ to obtain the twisted diagonal.

We can work component by component in the cuspidal unitary principal series (each component is closed and open).

Within a component with minimal K-types $\{\sigma_1, \ldots, \sigma_N\}$, we exclude $\pi_1 \otimes \pi_2^*$ with $\pi_1 \not\cong \pi_2$ by considering only the open set of reps including some $\sigma_j \otimes \sigma_j^*$. $\square$

- Is there a simpler argument? Shintaro will explain...
- What about the tempered dual of G/H?

The story so far...

In the group case of $(G\times G)/\Delta G$ (G linear reductive) it is possible to define a C^*-algebra starting from $L^2(G)$, viewed as a representation of $G\times G$.

- Can this be described as a convolution algebra?

I don't know, but there does exist a different, arguably more concrete, definition using C^*-ideas...

The C^*-algebraists use the work of Harish-Chandra, Langlands and others to write

$P = MAN$ cuspidal parabolic
τ = disc. series of M

$$C^*_r(G) \xrightarrow{\cong} \bigoplus_{[P,\tau]} C_0(\mathfrak{a}^*_P, \mathcal{K}(H_\tau))^{W_\tau}.$$

compact Hilbert space operators

Consider the following small modification:

$$\mathcal{E}(G) = \bigoplus_{[P,\tau]} C_0(\mathfrak{a}^*_P, H_\tau \otimes H^*_\tau)^{W_\tau}.$$

Hilbert-Schmidt operators

This is a C^*-module over the commutative C^*-algebra

$$\mathcal{A} = \bigoplus_{[P,\tau]} C_0(\mathfrak{a}^*_P)^{W_\tau}$$

of scalar C_0-functions on the tempered dual.

The Hilbert module $\mathcal{E}(G)$ carries a unitary representation of $G\times G$, and ...

our C^*-algebra for $(G\times G)/\Delta G$

Theorem The C^*-algebra subquotient of $C^*(G\times G)$ associated to the twisted diagonal (a locally closed set) is the intersection of the image of

$$C^*(G\times G) \longrightarrow \mathcal{L}(\mathcal{E}(G))$$

with the C^*-algebra ideal of compact operators on $\mathcal{E}$.

- Is it possible to define the commutative C^*-algebra $\mathcal{A}$ directly (without recourse to the Fourier transform / Plancherel decomposition)? It is a C_0-version of the commutant of the bi-regular representation...

A final result, which is not obvious, but also not difficult, using the Fourier transform picture of $C_r^*(G)$:

Theorem The C^*-algebra subquotient of $C^*(G\times G)$ associated to the twisted dual (a locally closed set) is Morita equivalent to $C_r^*(G)$.

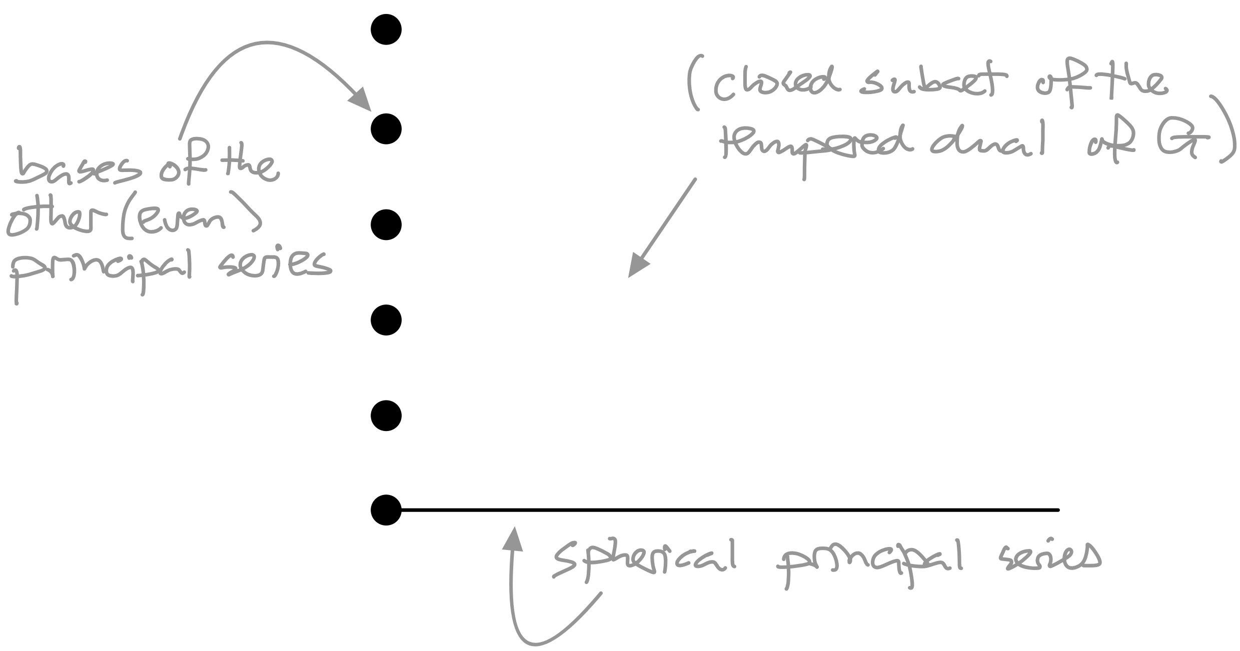
So it has the "correct" K-theory (as described by Connes & Kasparov).

One example beyond the group case

Consider $G/H = SL(2,\mathbb{C})/SL(2,\mathbb{R})$. This was studied very early on. It is a tempered symmetric space, in the sense that Toshi has described to us:

$$\begin{tikzcd} & C^*_r(G) \arrow[dr] & \\ C^*(G) \arrow[ur] \arrow[rr] & & \mathbb{B}(L^2(G/H)) \end{tikzcd}$$

This makes the dual a bit easier to understand (for representation-theory lightweights like me):



(Full disclosure: I did not actually define the tempered dual of any G/H!)

- Is there a "topological" or "geometric" construction of the discrete series in this case?
- In general, can we expect tempered symmetric spaces to be more accessible?

Thank You!

(and over to you, Shintaro)