RTNCG before 1950, lecture 4: Representations of noncompact groups

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CNRS & Institut Élie Cartan de Lorraine

AIM RTNCG – October 10, 2022

Previous lectures:

- Compact groups, **spectral theory**, Peter–Weyl theorem
- Locally compact abelian groups and **harmonic analysis**

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- Supporting characters for Episodes 2-3:

  Quantum physics (Hilbert space, spectral theory),

  Positive-definite functions (on abelian groups), Today:

Representation theory of noncompact groups

1939–1947

A tangled trail, continued

Papers by Gelfand et al. — 1939-1942

Operator algebras – Group theory

On one-parametrical groups of operators in a normed space

Dokl. Akad. Nauk SSSR **25** (1939) 713–718. Zbl. **22**:358

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A tangled trail, continued

George Mackey on our subject

A full-scale attack on extending the theory to the general locally compact case

began rather abruptly at the end of 1946.

Nevertheless three more or less unrelated but extremely important steps were made before 1945

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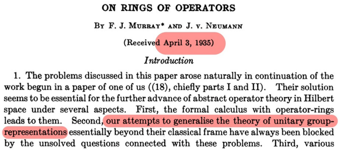
ON RINGS OF OPERATORS

BY F. J. MURRAY* AND J. V. NEUMANN

(Received April 3, 1935)

Introduction

1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various

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THE GROUP RING OF A LOCALLY COMPACT GROUP. I

BY I. E. SEGAL[1]

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated June 10, 1941

1. The theory of topological groups has an extensive development in the cases of locally compact[2] (l. c.) Abelian groups and compact groups. The theory of almost periodic functions on groups has proved helpful in investigating these groups, and is of interest in itself. However, there exist l. c. groups on which there are defined no almost periodic functions except for constant functions. It can therefore not be expected that these functions will be useful in investigating general l. c. groups.

In §2 we define for any l. c. group G a "group ring" $R(G)$. If G is finite $R(G)$ is the usual group ring over the field of complex numbers. $R(G)$ is

Late 1930s ideas to go beyond compact groups

- **Almost periodic functions**
- **Operator algebras**
- **Mystery third**

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- **Almost periodic functions**
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André Weil on his 1940 book (written 1935-1936)

My main hope was to open the way for a generalization beyond compact groups...

Not only did I not reach the promised land of infinite-dimensional representations, but I stopped before I had even a glimpse of it.

I lost courage when I saw that for finite-dimensional reps of noncompact simple groups, the matrix coefficients are not square-integrable.

Perhaps it took the prejudice-free mind of a physicist to go further...

Gruppenpest

Wigner's early use of representation theory in quantum mechanics

Eugene Wigner (1902–1995)

Gruppenpest

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- **1927 paper:** self-adjoint operators commuting with a finite group rep.
- Application to systems of n electrons, using the symmetric group $\mathfrak{S}_n$

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1939

- Fundamental paper on the inhomogeneous Lorentz group

Eugene Wigner (1902–1995)

Weyl's first paper on group theory and quantum mechanics

In quantum mechanics, one can ask two clearly distinct questions:

1. *How do I **find the linear operator** corresponding to a given physical quantity?*
2. ***Once I have the operator**, what **physical meaning** does it have?*

*To the **second question**, von Neumann has given a clear and far-reaching answer...*

- If system has built-in symmetries, **Hilbert space of possible states must carry representation**

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Heisenberg group?
Link between projective representations and central extensions: known from Schur (1907-1911)...
But I found no early mention of the Heisenberg group (under this or another name)

Wigner's 1939 paper

ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

By E. Wigner

(Received December 22, 1937)

Mathematical content

Description of the **unitary irreducible representations of** $\mathrm{SO}(3,1) \ltimes \mathbb{R}^4$

... or rather double cover

$SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$

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Physical relevance

We see thus[5] that there corresponds to every invariant quantum mechanical system of equations such a representation of the inhomogeneous Lorentz group. This representation, on the other hand, though not sufficient to replace the quantum mechanical equations entirely, can replace them to a large extent.

A. Previous treatments

The representations of the Lorentz group have been investigated repeatedly. The first investigation is due to Majorana,[7] who in fact found all representations of the class to be dealt with in the present work excepting two sets of representations. Dirac[8] and Proca[8] gave more elegant derivations of Majorana's results

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Acknowledgement. The subject of this paper was suggested to me as early as 1928 by P. A. M. Dirac who realised even at that date the connection of representations with quantum mechanical equations. I am greatly indebted to him also for many fruitful conversations about this subject, especially during the years 1934/35, the outgrowth of which the present paper is.

I am indebted also to J. v. Neumann for his help and friendly advice.

Wigner's 1939 paper

The difference between the present paper and that of Majorana and Dirac lies—apart from the finding of new representations—mainly in its greater mathematical rigor. Majorana and Dirac freely use the notion of infinitesimal operators and a set of functions to all members of which every infinitesimal operator can be applied. This procedure cannot be mathematically justified

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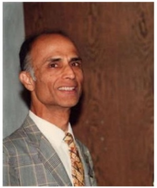

Harish-Chandra at the Institute in 1947

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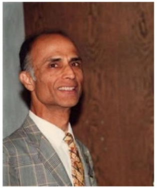

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Classification of unitary representations according to von Neumann and Murray[10]

According to Murray and von Neumann, if the original representation was factorial, all representations into which it can be decomposed will be factorial also. Thus every representation is equivalent to a sum of factorial representations, part of which is "normal," the other part "pathological."

It will turn out again that the inhomogeneous Lorentz group has no pathological representations. Thus this assumption of Majorana and Dirac also will be justified a posteriori. Every unitary representation of the inhomogenous Lorentz group can be decomposed into normal irreducible representations. It

Wigner's 1939 paper

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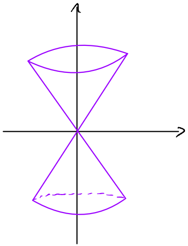
Representations of $\mathrm{SO}(3,1) \ltimes \mathbb{R}^4$

- Action of $\mathrm{SO}(3,1)$ on $(\mathbb{R}^4)^*$ (**physical momenta**)
- **Dirac: Wave functions should be vector-valued functions on $(\mathbb{R}^4)^*$**
- **Space** of allowed wave functions should be **invariant**



Construction of representations

Wigner's 1939 paper

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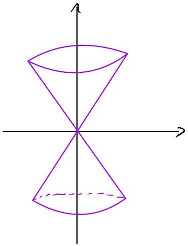

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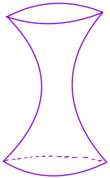

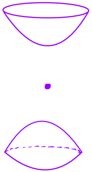

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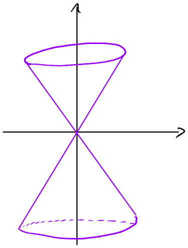

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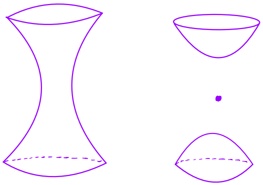

The pure translations form an invariant subgroup of the whole inhomogeneous Lorentz group and Frobenius' method[15] will be applied in Section 6 to build up the representations of the whole group out of representations of the subgroup,

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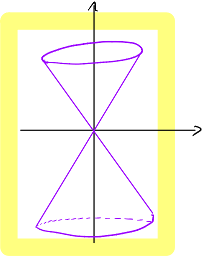

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The momentum vectors of the
1*st class* are time-like,
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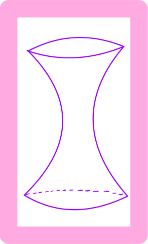

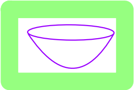

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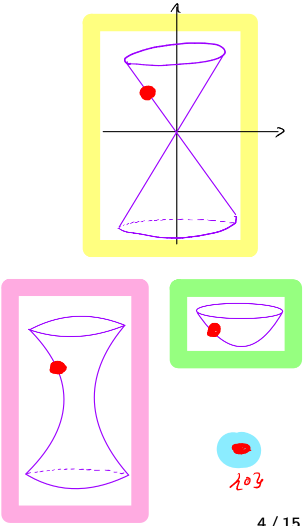
Reps of stabilizers



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"Little groups":

$$\left.\begin{array}{l} SO(3) \\ SO(2) \ltimes \mathbb{R}^2 \end{array}\right\} \text{reps. known}$$

$$\left.\begin{array}{l} SO(3,1) \\ SO(2,1) \end{array}\right\} \text{reps ??}$$

The Gelfand–Raikov theorem

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- **The theorem:** $\boxed{\text{for any } g \neq e, \text{ there exists a unitary irreducible rep. } U \text{ such that } U(g) \neq 1}$

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Krein (1939)

Kolmogorov for probability

The Gelfand–Raikov theorem

- **Krein (1939)** defines $f : G \to \mathbb{C}$ to be positive-definite when

  Hermitian matrix $(f(g_i g_j^{-1}))_{1\leqslant i,j\leqslant n}$ is non-negative for all n and all $g_1, \ldots, g_n \in G$.

- **First observation of Gelfand and Raikov:**

  - If $U : G \to \mathscr{B}(H)$ is a unitary rep. then $g \mapsto \langle v, U(g)v\rangle$ is positive-definite for any v.
  - If f is continuous pos-def, then it arises as a **diagonal matrix coefficient** of a unitary rep.

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Proof (2^d pt): take $\boxed{\{\varphi : G \to \mathbb{C} \text{ with finite support}\}}$ & inner product $\boxed{\langle \varphi, \psi \rangle_f = \sum f(a^{-1}b)\, \varphi(a)\, \overline{\psi(b)}}$

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$\leadsto$ Complete into $\boxed{\text{Hilbert space } L^2_f}$; then $\boxed{\text{left translations}}$ induce G-rep. π

The Gelfand–Raikov theorem

- **Krein (1939)** defines $f : G \to \mathbb{C}$ to be positive-definite when

  Hermitian matrix $(f(g_i g_j^{-1}))_{1 \leqslant i,j \leqslant n}$ is non-negative for all n and all $g_1, \ldots, g_n \in G$.

- **First observation of Gelfand and Raikov:**

  - If $U : G \to \mathscr{B}(H)$ is a unitary rep. then $g \mapsto \langle v, U(g)v \rangle$ is positive-definite for any v.
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$\leadsto$ Complete into $\boxed{\text{Hilbert space } L^2_\varphi}$; then $\boxed{\text{left translations}}$ induce G-rep. π

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- **Irreducible unitary** representations correspond to **elementary continuous pos-def** functions

f elementary: If $f = f_1 + f_2$ with f_1, f_2 positive-definite

then $f_1 = \alpha f$, $f_2 = (1-\alpha) f$ $(\alpha > 0)$

Geometry of positive-definite functions

- Positive-definite functions with $f(e) \leqslant 1$ form a **convex set**.
- **Extreme points** are: 0 and the **elementary continuous pos-def functions with** $f(e) = 1$

The Gelfand–Raikov proof

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Connection with linear functionals on $L^1(G)$

- Consider $L^1(G)$ with **convolution product** and **involution** $f \mapsto \overline{f(g^{-1})}\Delta(g)^{-1}$.
- If f is continuous pos-def, then **linear functional** $L(\varphi) = \int_G f\varphi$ **is positive**: $L(\varphi * \tilde{\varphi}) \geqslant 0$ for all φ
- **Every positive linear functional on $L^1(G)$** arises in this way.
- Can consider **weak topology** on $L^1(G)$, and thus on continuous pos-def functions.

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- Positive-definite continuous functions with $f(e) \leqslant 1$ form a **convex set** $\mathscr{P}$.
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- **Krein–Milman:** $\mathscr{P}$ is the **weakly closed convex hull** of its extreme points

The Gelfand–Raikov paper

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Conclusion for the original problem

- Enough to show: for $g \neq e$, there exists **elementary continuous pos-def** f **s.t.** $f(g) \neq 1$.
- Without asking for **elementary**, this is easy:
  - Take left regular representation L on $L^2(G)$, and $\varphi \in \mathscr{C}_c(G)$ with support near g and norm 1,
  - Then $f(x) = \langle \varphi, L(x)\varphi \rangle_{L^2(G)}$ has desired property.

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  - Then $f(x) = \langle \varphi, L(x)\varphi \rangle_{L^2(G)}$ has desired property.
- This f is **close to a finite combination of elementary** continuous pos-def functions.

Gelfand, Godement, Segal

All working towards a general representation theory in the early 1940s

THE GROUP RING OF A LOCALLY COMPACT GROUP. I

BY I. E. SEGAL[1]

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated June 10, 1941

1. The theory of topological groups has an extensive development in the cases of locally compact[2] (l. c.) Abelian groups and compact groups. The theory of almost periodic functions on groups has proved helpful in investigating these groups, and is of interest in itself. However, there exist l. c. groups on which there are defined no almost periodic functions except for constant functions. It can therefore not be expected that these functions will be useful in investigating general l. c. groups.

In §2 we define for any l. c. group G a "group ring" $R(G)$. If G is finite $R(G)$ is the usual group ring over the field of complex numbers. $R(G)$ is defined in terms of Haar measure on G. It is known that there exists on any l. c. group a Haar measure, and this assures the existence of a non-trivial group ring. In this note we present an account of the ideal theory of a group ring for the case in which the group is either l. c. Abelian or compact. However, the conclusions of our principal theorems (Theorems 1 and 2) have meaning, and may well be true, in the case that the group in question is an arbitrary l. c. group.

Gelfand, Godement, Segal

All working towards a general representation theory in the early 1940s

This 1941 algebra:

Take $L^1(G)$ and add unit

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All working towards a general representation theory in the early 1940s

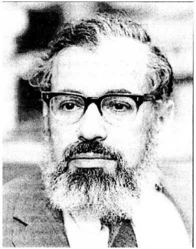
The twin papers of 1947

THE GROUP ALGEBRA OF A LOCALLY COMPACT GROUP[1]

BY
I. E. SEGAL

IRREDUCIBLE REPRESENTATIONS OF OPERATOR ALGEBRAS

I. E. SEGAL



(Coins the term " C*-algebra ")

Gelfand, Godement, Segal

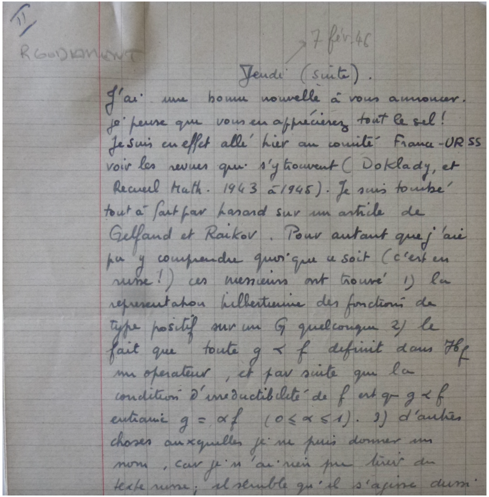
All working towards a general representation theory in the early 1940s

Godement's thesis work, 1943-1946:

General theory of spherical functions

Gelfand, Godement, Segal

All working towards a general representation theory in the early 1940s



11

R GODEMENT

7 fév. 46

Jeudi (suite).

J'ai une bonne nouvelle à vous annoncer. Je pense que vous en apprécierez tout le sel ! Je suis en effet allé hier au comité France-URSS voir les revues qui s'y trouvent (Doklady, et Recueil Math. 1943 à 1945). Je suis tombé tout à fait par hasard sur un article de Gelfand et Raikov. Pour autant que j'ai pu y comprendre quoi que ce soit (c'est en russe !) ces messieurs ont trouvé 1) la représentation hilbertienne des fonctions de type positif sur un G quelconque 2) le fait que toute $g \prec f$ définit dans $\mathcal{H}_f$ un opérateur, et par suite que la condition d'irréductibilité de f est que $g \prec f$ entraîne $g = \alpha f$ $(0 \leq \alpha \leq 1)$. 3) d'autres choses auxquelles je ne puis donner un nom, car je n'ai rien pu tirer du texte russe ; il semble qu'il s'agisse d'un

Godement to Cartan, Feb. 1946
(thanks to Christophe Eckes)

George Mackey on our subject

A full-scale attack [...] began rather abruptly at the end of 1946

[...] in 1947, more than six contributions from six authors

Papers from 1947:

The Lorentz group everywhere

George Mackey on our subject

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Papers from 1947:

- *Unitary representations of the Lorentz group*, by Gelfand and Naimark,
- *Irreducible unitary representations of the Lorentz group*, by Bargmann,
- *Infinite irreducible representations of the Lorentz group*, by Harish-Chandra,
- *Unitary representations of the group of transformations of the straight line,* by Gelfand and Naimark,
- *The group algebra of a locally compact group*, by Segal,
- *Irreducible representations of operator algebras*, by Segal,
- *Sur les relations d'orthogonalité de Bargmann*, by Godement.

(with M. A. Najmark)

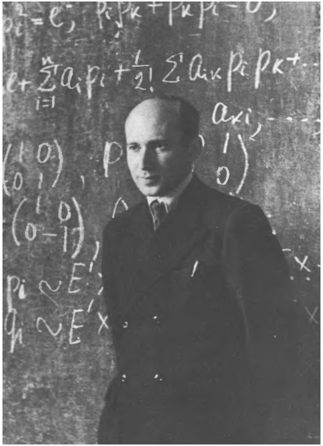
Unitary representations of the Lorentz group

Izv. Akad. Nauk SSSR, Ser. Mat. **11** (1947) 411–504. Zbl. **37**:153

In this paper all unitary irreducible representations of the unimodular complex group of second order are determined; This group is locally isomorphic to the Lorentz group.

It is proved also that an arbitrary unitary representation can be decomposed into these irreducible representations.

As far as possible, the theorems are proved in such a way that similar proofs can be carried out for all complex semisimple groups.



Gelfand–Naimark on $SL(2, \mathbb{C})$

(with M. A. Najmark)

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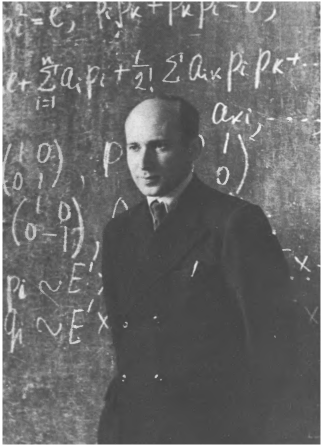

Gelfand–Naimark on $\mathrm{SL}(2, \mathbb{C})$

Structure theory:

§1. Some subgroups of the group G

§2. Some relations between the group G and the subgroups H, K, Z, Z, D

§3. Some relations between integrals over the group G and over the subgroups K, H, Z, Z, D

§4. Cosets of G by Z and by K

(Z = upper triangular, D = diagonal)

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§5. The principal series of irreducible representations of the group G

§6. The trace of the representation of the principal series

In our representations, to each element g of the Lorentz group we associate a unitary operator U_g in an infinite-dimensional space; this operator has no trace in the usual sense. However, an operator $\int_Q U_g d\mu(g)$ for compact Q has a trace. Hence, we can define the trace of the operator U_g and the character of the representation [←6, formula (94)].

Formulas of representation theory are usually quite elegant. Therefore, the authors have tried not only to give the existence proofs but to write down each result in the final form.

Normed rings reappear here...

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§7. Decomposition of the regular representation of the group G into irreducible representations; analog of the Plancherel theorem

Further, we determine the decomposition of the regular representation into irreducible representations. It turns out that the decomposition of the regular representation does not include all irreducible representations (as in the case of commutative and compact groups) but only the representations of the so-called "principal series". This fact is not due to complications related to the set theory but is quite "classic". We obtain also an analog of the Plancherel formula.

Gelfand–Naimark on $\mathrm{SL}(2,\mathbb{C})$

Structure theory:

§1. Some subgroups of the group *G*

§2. Some relations between the group *G* and the subgroups *H*, *K*, *Z*, Z, *D*

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8. The complementary series of irreducible representations of the group *G*

Theorem 10. *If $x(g) = (x_1^* \times x_1)(g)$ and $x_1(g) \in R'$ then the operator $U_{\rho_1;x}$ has a trace and this trace is given by formulas* (225) *and* (226).

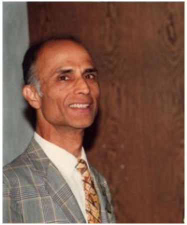
This theorem, as in §6, immediately implies:

Theorem 11. *For an arbitrary integrable function $x(g)$ an operator $U_{\rho_1;x}$ of the irreducible representation of the complementary series is compact.*

Theorem 12. *Representatios of the complementary series that correspond to different ρ_1 from the interval $0 < \rho_1 < 2$ are not equivalent to each other and they are not equivalent to representations of the principal series.*

§9. Decomposition of an arbitrary unitary representation of the group *G* into representations of the principal and complementary series

Harish-Chandra, young physicist

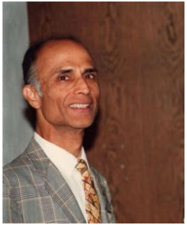


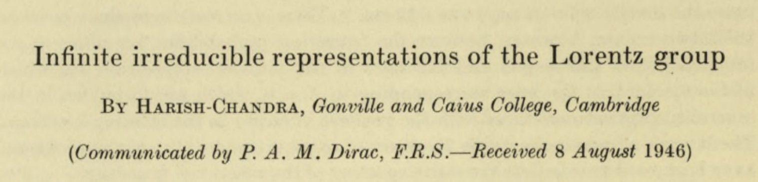
Harish-Chandra at the Institute in 1947

In Princeton I learned that not every function is analytic.

After that I couldn't be a physicist anymore...

Harish-Chandra, young physicist



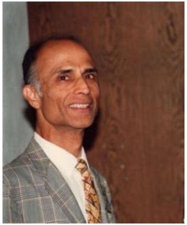


Infinite irreducible representations of the Lorentz group

By Harish-Chandra, *Gonville and Caius College, Cambridge*

(*Communicated by P. A. M. Dirac, F.R.S.—Received* 8 *August* 1946)

Harish-Chandra, young physicist



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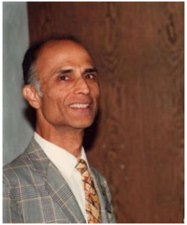
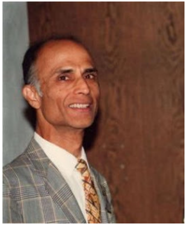
Further, as pointed out by Dirac (1945), it is possible to use expinors to describe the transformation properties of the wave function of a spinning particle. In a theory based on these expinors it is possible to make the charge density positive definite for particles of integral spin or the energy density positive definite for particles of half-integral spin, in contradistinction with the results of the existing theory (see Pauli 1940). This is made possible by the circumstance that infinite unitary representations of the Lorentz group exist for both integral and half-integral spins. By imposing subsidiary conditions on the wave function, it can be

Harish-Chandra, young physicist



Consider now an irreducible representation of the proper Lorentz group. The matrices I^{kl} then form an irreducible set. Denote the representation space by $\mathfrak{R}$. Reduce this space with respect to the subgroup δ_3 consisting of spatial rotations only. Now δ_3 is a compact group, and it is well known that every representation of a compact group is completely reducible into a direct sum of irreducible representations each of which is unitary and of a finite degree (see Pontrjagin 1939, p. 205). Now every irreducible representation of δ_3 is characterized by a number $k \geqslant 0$ which is integral or half-integral. Thus $\mathfrak{R}$ is decomposed into a direct sum of subspaces $\mathfrak{R}_k$ which are all irreducible with respect to δ_3. It will be assumed† that in this decomposition there occurs at most only one subspace $\mathfrak{R}_k$ for any particular value of k.

Harish-Chandra, young physicist



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† *Note added in proof.* It is possible to avoid this assumption. The proof is then somewhat more complicated but the final result is the same.

Bargmann on $\mathrm{SL}(2, \mathbb{R})$

Valentine Bargmann (1980–1989)

Bargmann on $\mathrm{SL}(2, \mathbb{R})$

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1947 paper:

- Studies reps. of $\mathrm{SO}(3, 1)$, $\mathrm{SO}(2, 1)$, and double cover $\mathrm{SL}(2, \mathbb{R})$

Bargmann on $\mathrm{SL}(2, \mathbb{R})$

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- Studies reps. of $\mathrm{SO}(3,1)$, $\mathrm{SO}(2,1)$, and double cover $\mathrm{SL}(2, \mathbb{R})$
- **First steps to approach representation theory:**

Lie algebra representations

&

eigenvalues of Lie algebra Hermitian operators

Bargmann on $\mathrm{SL}(2,\mathbb{R})$

5 f. *Construction of the infinitesimal representations of* $\mathfrak{S}$. The one parameter subgroup $a(t) = \exp(t\chi_0)$ is a *compact* Abelian group. By comparison of (4.11) and (4.12) it is seen that $\exp(4\pi\chi_0) = e$. It follows that $U_t = \exp(-itH_0)$ has a *pure point spectrum*, i.e., there exists a complete orthonormal system of vectors g_n in $\mathfrak{H}$ such that[15]

$$U_t\, g_n = e^{-i\lambda_n t}\, g_n\,; \qquad H_0\, g_n = \lambda_n\, g_n\,. \tag{5.18}$$

Since $a(4\pi) = e$, the proper values λ_n may be *integral* or *half integral*. We may derive (5.18) directly from Stone's Theorem, or we may use the fact that a unitary representation of any compact group may be decomposed into finite-dimensional irreducible parts (which, for Abelian groups, are one-dimensional and of the form (5.18)). (Cf. [Wigner, p. 194]).

Choose one of the proper vectors of H_0. Denote it by g ($\|g\| = 1$) and let λ be the corresponding proper value, so that $H_0 g = \lambda g$. By (5.17) $g \in \mathfrak{A}$, and hence

H_0 = self-adjoint op. on $\mathcal{H}$ from $\mathrm{Lie}(SO(2))$

Look at spectrum of H_0

and action of $\underline{\mathfrak{sl}}(2,\mathbb{C})$ on eigenvectors

Bargmann on $\mathrm{SL}(2,\mathbb{R})$

5 f. *Construction of the infinitesimal representations of $\mathfrak{S}$.* The one parameter subgroup $a(t) = \exp(t\chi_0)$ is a *compact* Abelian group. By comparison of (4.11) and (4.12) it is seen that $\exp(4\pi\chi_0) = e$. It follows that $U_t = \exp(-itH_0)$ has a *pure point spectrum*, i.e., there exists a complete orthonormal system of vectors g_n in $\mathfrak{H}$ such that[15]

$$U_t g_n = e^{-i\lambda_n t} g_n \, ; \qquad H_0 g_n = \lambda_n g_n \, . \tag{5.18}$$

Since $a(4\pi) = e$, the proper values λ_n may be *integral* or *half integral*. We may derive (5.18) directly from Stone's Theorem, or we may use the fact that a unitary representation of any compact group may be decomposed into finite-dimensional irreducible parts (which, for Abelian groups, are one-dimensional and of the form (5.18)). (Cf. [Wigner, p. 194]).

Choose one of the proper vectors of H_0. Denote it by g ($\|g\| = 1$) and let λ be the corresponding proper value, so that $H_0 g = \lambda g$. By (5.17) $g \in \mathfrak{A}$, and hence

H_0 = self-adjoint op. on $\mathcal{H}$ from Lie(SO(2))

Look at spectrum of H_0

and action of $\underline{\mathfrak{sl}}(2,\mathbb{C})$ on eigenvectors

5g. *Classification of the possible representations. I. The continuous class.* Assume first that all vectors of the series $F^s g$, $G^s g$ are different from zero. Then all numbers $\lambda \pm s$ are proper values of H_0, and are either all integral or half integral. Moreover, all numbers ρ_s and σ_s are *positive*. (Cf. (5.26)) It follows

II. The discrete class.
(1) Assume now that for some positive s $F^s g = 0$. This will also hold for all succeeding s. If $h + 1$ ($h \geqq 0$) is the *smallest* integer for which this occurs then the vectors $g, \cdots, F^h g$ are all different from zero, and it again follows that all vectors in the series may be obtained from $F^h g$ by applying a power of G to it,

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§6. Representations of the continuous class, C_q^0 ($q \geq \frac{1}{4}$)

It is easy to construct unitary representations of $\mathfrak{L}$ (and hence of $\mathfrak{S}$) as long as they are not required to be irreducible. For example, one may choose in the Euclidean space of the three variables x^0, x^1, x^2, the manifold $\mathfrak{M}$ (invariant under Lorentz transformations) defined by the equation $g_{kl}x^k x^l = d(=\text{const.})$, and consider the transformations $y = ax$ which are induced on $\mathfrak{M}$ by the Lorentz transformations of the x^k. (Depending on the value of d, $\mathfrak{M}$ is a hyperboloid, of one sheet ($d > 0$), or of two sheets ($d < 0$), or it is a cone ($d = 0$), the light-cone of special relativity). A volume element which is invariant with respect to the standard transformations $T^0(a)f = f(a^{-1}x)$ is readily defined on $\mathfrak{M}$. Therefore the $T^0(a)$ are unitary operators on the Hilbert space of all square-integrable functions over $\mathfrak{M}$, and they furnish a representation of $\mathfrak{L}$ since $T^0(a)T^0(b) = T^0(ab)$.

A simple analysis shows that these representations are *reducible*. It turns out that the operators $T^0(a)$ are particularly simple if $\mathfrak{M}$ is the light-cone. In what follows we shall show that a further reduction leads to *irreducible* representations (of the continuous class C_q^0).

§9. Representations of the discrete classes D_k^+ and D_k^- [16]

9a. *The multiplier representations* T_l. We consider here as the manifold $\mathfrak{M}$ the open unit circle in the complex plane ($z\bar z < 1$), i.e., the manifold $\mathfrak{M}^*$ of §4e. On $\mathfrak{M}$ the group $\mathfrak{S}$ is realized by the conformal transformations of the unit circle onto itself. We have

$$z' = az = \frac{\bar\alpha z + \bar\beta}{\beta z + \alpha} \quad (\alpha\bar\alpha - \beta\bar\beta = 1) \qquad (a \in \mathfrak{S},\ z \in \mathfrak{M}) \tag{9.1}$$

(cf. (4.23)). A multiplier of (1) is given by

$$\mu(a, z) = \alpha + \beta z \tag{9.2}$$

(cf. (4.25)). We observe that

$$\frac{dz'}{dz} = \mu(a, z)^{-2}\ 1 - z'\bar z' = |\mu(a, z)|^{-2}(1 - z\bar z) \qquad (z' = az). \tag{9.3}$$

[16] The construction described in this section is closely related to Dirac's construction of the expansor representation [Dirac 2]. Cf. also the Appendix to Part II of this paper.

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- Finds **explicit formulas for matrix coefficients**
- Studies **asymptotic behavior of matrix coefficients**
- Proves $L^2(G)$ **densely spanned** by **discrete series matrix elts.** + **wave packets** of princ. series

Godement's ideas on discrete series

1947 CRAS notes:

- Notion of **square-integrable representation**
- Abstract proof of **Bargmann's orthogonality relations**

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- **Square-integrability modulo center**, in case $Z(G)$ is finite
- Taken up by Harish-Chandra, reductive-specific proofs

Promised this series would stop somewhere around 1947...

Papers from 1947:

- *Unitary representations of the Lorentz group*, by Gelfand and Naimark,
- *Irreducible unitary representations of the Lorentz group*, by Bargmann,
- *Infinite irreducible representations of the Lorentz group*, by Harish-Chandra,
- *Unitary representations of the group of transformations of the straight line,* by Gelfand and Naimark,
- *The group algebra of a locally compact group*, by Segal,
- *Irreducible representations of operator algebras*, by Segal,
- *Sur les relations d'orthogonalité de Bargmann*, by Godement.

8

Direct integral decompositions

Beginnings and application to representation theory:

- **Von Neumann 1938-1939**: “rings of operators” are key to “continuous sum” decompositions
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Where representation theorists still can't escape operator algebras:

- **Plancherel formula** (1950) (Segal, Mautner, Godement, ...)
- Notion of **Type I groups and C^*-algebras**, and "liminal/CCR" (Mautner, Kaplansky)

Induced representations

George Mackey
(1916–2006)

Mackey on the general notion of induced representation

In 1949 I stumbled upon a general way of constructing unitary representations which includes many of the constructions used by Gelfand, Neumark and Bargmann... I use the phrase "stumbled upon" because I was not seeking such a construction at all. I had just become aware of the 1930 paper of Stone...

After 1950, depth and breadth in study of particular classes of groups

1. Compact groups, connections to spectral theory and with harmonic analysis
2. Abelian locally compact groups: harmonic analysis, almost periodic functions and topology
3. Beginnings of operator algebras: group representations and quantum mechanics behind the curtain
4. Beginnings of noncompact group representations, connections with physics and operator algebras