RTNCG before 1950, lecture 3: Beginnings of operator algebras

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AIM RTNCG – October 3, 2022

Previous lectures:

- Compact groups, **spectral theory**, Peter–Weyl theorem
- Locally compact abelian groups and **harmonic analysis**

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  Wiener's theorem on Fourier series

($f: \mathbb{R} \to \mathbb{C}$ periodic, $f \neq 0$ everywhere)

Fourier series for f absolut. cvg
$\Downarrow$
Fourier series for $1/f$ absolut. cvg

Next two lectures:

Beginnings of operator algebras
&
Representation theory of locally compact groups

A tangled trail

Papers by Gelfand et al. — 1939-1942

Operator algebras – Group theory

On one-parametrical groups of operators in a normed space

Dokl. Akad. Nauk SSSR **25** (1939) 713–718. Zbl. **22**:358

On normed rings

Dokl. Akad. SSSR **23** (1939) 430–432. Zbl. **21**:294

To the theory of normed rings. II.
On absolutely convergent trigonometrical series and integrals

Dokl. Akad. Nauk SSSR **25** (1939) 570–572. Zbl. **22**:357

To the theory of normed rings. III.
On the ring of almost periodic functions

Dokl. Akad. Nauk SSSR **25** (1939) 573–574. Zbl. **22**:357

(with D. A. Rajkov)

On the theory of characters of commutative topological groups

Dokl. Akad. Nauk SSSR **28** (1940) 195–198. Zbl. **24**:120

Normierte Ringe[1]

Mat. Sb., Nov. Ser. **9** (51) (1941) 3–23. Zbl. **24**:320

Zur Theorie der Charaktere
der Abelschen topologischen Gruppen

Mat. Sb., Nov. Ser. **9** (51) (1941) 49–50. Zbl. **24**:323

Ideale und primäre Ideale in normierten Ringen

Mat. Sb., Nov. Ser. **9** (51) (1941) 41–48. Zbl. **24**:322

(with D. A. Rajkov)

Irreducible unitary representations of locally bicompact groups

Mat. Sb. **13** (55), 301–316 (1942)

(with M. A. Najmark)

On the embedding of normed rings into the ring of operators in Hilbert space

Mat. Sb., Nov. Ser. **12** (54) (1942) 197–213. Zbl. **60**:270

A tangled trail

ON RINGS OF OPERATORS

By F. J. Murray* and J. v. Neumann

(Received April 3, 1935)

Introduction

1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various

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(Is this really Segal?)

THE GROUP RING OF A LOCALLY COMPACT GROUP. I

By I. E. Segal[1]

Department of Mathematics, Harvard University

Communicated June 10, 1941

1. The theory of topological groups has an extensive development in the cases of locally compact[2] (l. c.) Abelian groups and compact groups. The theory of almost periodic functions on groups has proved helpful in investigating these groups, and is of interest in itself. However, there exist l. c. groups on which there are defined no almost periodic functions except for constant functions. It can therefore not be expected that these functions will be useful in investigating general l. c. groups.

In §2 we define for any l. c. group G a "group ring" $R(G)$. If G is finite $R(G)$ is the usual group ring over the field of complex numbers. $R(G)$ is

Today's subject:

Operator algebras: prehistory and Gelfand's work

Hermann Weyl on Hilbert's spectral theory

The story would have been dramatic enough had it ended here.

But then a sort of miracle happened: the spectral theory in Hilbert space

was discovered to be the adequate mathematical instrument of the new quantum physics...

André Weil on his 1926 visit to Göttingen

As I found out much later, physics was thriving in Göttingen at the time:

Quantum mechanics was in the incubation stage.

It is remarkable that I had not the slightest inkling of this while I was there.

1913 book:

- Consider Hilbert space $E = \ell^2(\mathbb{Z})$ and **algebra** $\mathscr{L}(E)$.
- Modern definition of **spectrum of an element** $A \in \mathscr{L}(E)$

F. Riesz reworks Hilbert's spectral theory

1913 book:

- Consider Hilbert space $E = \ell^2(\mathbb{Z})$ and **algebra** $\mathscr{L}(E)$.
- Modern definition of **spectrum of an element** $A \in \mathscr{L}(E)$
- **Functional calculus** for symmetric operators:

  If $A \in \mathscr{L}(E)$ symmetric,
  can define $f(A)$ for any semi-continuous $f : \mathbb{R} \to \mathbb{R}$.

- Very simple approach to Hilbert's spectral theorem

Frygies Riesz
(1880–1956)

F. Riesz reworks Hilbert's spectral theory

Riesz's proof of Hilbert's spectral theorem:

- Begin with a **symmetric bounded operator** A on $\mathscr{L}(E)$.
- **Functional calculus** defines $f(A)$ for $f = 1_{]-\infty,\lambda]}$

  $\rightsquigarrow$ get operators A_λ, $\lambda \in \mathbb{R}$
- Riesz proves

$$\langle Ax, y \rangle = \int_{\mathbb{R}} \lambda \, d\langle A_\lambda x, y \rangle$$

$$A = \int_{\mathbb{R}} \lambda \, dA_\lambda$$

- **Spectrum of A:** points around which $\lambda \mapsto A_\lambda$ isn't locally constant.

Frygies Riesz
(1880–1956)

1927–1928 papers:

- Notion of **abstract Hilbert space**
- Notion of **unbounded self-adjoint operator**, and **spectral theorem.**

Stone's formulation
(1932 book)

von Neumann's spectral theory

1927–1928 papers:

- Notion of **abstract Hilbert space**
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Stone's formulation
(1932 book)

- **Resolution of the identity** in H:
  - Family $(E_\lambda)_{\lambda \in \mathbb{R}}$ of projections
  - Request $\lambda \mapsto E_\lambda$ increasing, right-continuous, goes to 0 as $\lambda \to -\infty$ and 1 as $\lambda \to +\infty$.
- Then **1-1 correspondence**

Unbounded self-adjoint ops. on H $\leftrightarrow$ Resolutions of the identity in H.

Resolution $(E_\lambda)_{\lambda \in \mathbb{R}}$ $\leftrightarrow$ unique self-adjoint A s.t. $\langle Au, u\rangle = \int_{\mathbb{R}} \lambda \langle dE_\lambda u, u\rangle$

von Neumann's spectral theory

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von Neumann's spectral theory – normal operators

Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren.

Von

J. v. Neumann in Berlin

1929 paper on normal operators:

- Consider the ring $\mathcal{B}$ of bounded operators on a Hilbert space H
- View this as an algebra, with adjunction operation $*$
- Will study **$*$-stable subalgebras** of $\mathcal{B}$

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Emmy Noether
(1882-1935)

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- **Weakly closed $*$-subalgebra:**

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- Must ask for subalgebras to be "closed", but several possible topologies
- **Weakly closed $*$-subalgebra:** **commutative $\iff$ generated by a single normal operator**

1929 paper, part I

- For any subset $\mathcal{M} \subset \mathcal{B}$, define

$$\mathrm{R}(\mathcal{M}) = \textbf{weak closure} \text{ of the algebra generated by } \mathcal{M}$$

$$\mathcal{M}' = \text{set of } A \in \mathcal{B} \text{ such that } A \text{ and } A^* \text{ commute with all of } \mathcal{M}$$

- Study **relation between** $\mathrm{R}(\mathcal{M})$ **and** $\mathcal{M}''$: they are **equal when** $1 \in \mathcal{M}$.
- Comments on **relationship with group representation theory**

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1929 paper, part II

- **Spectral theorem for normal operators**

Rings of operators

ANNALS OF MATHEMATICS
Vol. 37, No. 1, January, 1936

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(Received April 3, 1935)

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1) Natural sequel to 1929
2) Group representations
3) Quantum mechanics
4) Abstract algebras

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- Question: what are the $*$-subalgebras of $\mathcal{L}(H)$ that contain 1 and are **weakly closed**?
- Name "von Neumann algebra" appears in 1954 (Dixmier and Dieudonné)

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- Question: what are the $*$-subalgebras of $\mathcal{L}(H)$ that contain 1 and are **weakly closed**?
- Name "von Neumann algebra" appears in 1954 (Dixmier and Dieudonné)
- Huge series of papers: factors, type I-II-III, trace, direct integrals...

Stone and boolean algebras

- Spectral theorem and **projection-valued measures**

Marshall Stone
(1903–1989)

Stone and boolean algebras

- Spectral theorem and **projection-valued measures**

For any Borel subset $E \subset \mathbb{R}$, projection $P_E \in \mathcal{L}(H)$

$$\begin{cases} P_E P_F = P_{E \cap F} \\ P_E + P_F = P_{E \cup F} \quad (E, F \text{ disjoint}) \\ P_\emptyset = 0, \quad P_H = 1 \end{cases}$$

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Stone 1936:

- **Boolean algebra**: algebra in which every element is idempotent

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Stone 1936:

- **Boolean algebra**: algebra in which every element is idempotent
- A crucial example:
  - X locally compact space;
  - A: algebra generated by 1_Ω, $\Omega =$ open or negligible
  - Then **ideals in $A \leftrightarrow$ closed subsets of X.**

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An aside on Banach spaces and algebras

The Lwów school, 1922–1940

Banach spaces, functional analysis
(e.g. Banach's 1932 book)

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Nagumo 1936:

Notion of **Banach algebra**

Mazur 1938:

If A is a **normed unital algebra** over $\mathbb{C}$ and is a **field**, then $A = \mathbb{C} \cdot 1$

Spectral theory of Hilbert and Von Neumann

Riesz's functional calculus (1913)

Peter–Weyl theorem (1926)

Convolution as abstract operation

Algebras of operators (Von Neumann 1929–1939)

Abstract Banach algebras (Nagumo 1936)

Ideals in operator algebras (Stone 1936)

Mazur's theorem (1938)

Duality for abelian l.c. groups (1934)

Almost periodic functions (1934-1936)

Wiener's theorem (1932)

Positive-definite functions (Riesz 1933)

Enter Gelfand

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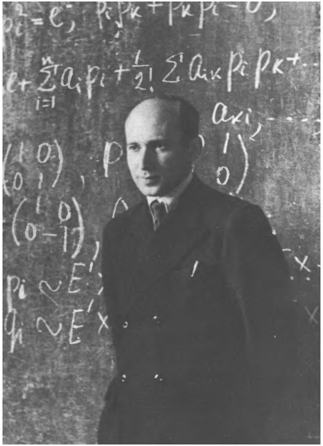

Israel M. Gelfand (1913–2009)

Gelfand and commutative algebras

1939 note (details 1941)

- Consider **commutative Banach unital algebra** R

Gelfand and commutative algebras

1939 note (details 1941)

- Consider **commutative Banach unital algebra** R
- Main object: the set $\mathfrak{M}$ of **maximal ideals of** R
  - They're all closed, and every nontrivial ideal is contained in a maximal ideal.
  - If $M \subset R$ max. ideal, then have **canonical isomorphism** $R/M \simeq \mathbb{C}$.

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  - They're all closed, and every nontrivial ideal is contained in a maximal ideal.
  - If $M \subset R$ max. ideal, then have **canonical isomorphism** $R/M \simeq \mathbb{C}$.
- Given $x \in R$, can associate a number $x(M)$ to any maximal ideal. Gives map

$$R \to \{\text{functions on } \mathfrak{M}\}$$

- Definition of topology on $\mathfrak{M}$ making these functions continuous, and $\mathfrak{M}$ compact
- Condition for $R \to \mathscr{C}(\mathfrak{M})$ to be an isomorphism.

Gelfand and commutative algebras

1939–1941

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- Notion of **holomorphic functional calculus on** R
- Application: if $\mathfrak{M}$ disconnected, then R splits as direct product

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Proof of Wiener theorem (1939 note)

- If $x \in R$, then: x **invertible in** $R \iff$ **there's no maximal ideal containing** x

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- **Maximal ideals of R:** sets $M_{t_0} = \{f \in R \ : \ f(t_0) = 0\}$, $t_0 \in (0, 2\pi)$.
- f **invertible in** A $\iff$ f contained in none of the ideals M_{t_0}

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- f **invertible in** A $\iff$ f contained in none of the ideals M_{t_0} $\iff$ f **vanishes nowhere**

Gelfand's school, 1939–1950

(With Raikov, Shilov)

(With Naimark)

(With Raikov, Naimark)

Gelfand's school, 1939–1950

Commutative C*-algebras Locally compact abelian groups	Abstract C*-algebras	Representations of noncompact groups
(With Raikov, Shilov)	(With Naimark)	(With Raikov, Naimark)
1939–1946: 10 papers	1942 : Gelfand–Naimark thm	

The Gelfand–Naimark theorem

Normed ring: Banach algebra with unit

A normed ring R will be called an $*$-ring if to every $x \in R$ there corresponds an element $x^* \in R$ satisfying the following conditions *:

1′. $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$;
2′. $x^{**} = x$;
3′. $(xy)^* = y^* x^*$;
4′. $\|x^* x\| = \|x^*\| \cdot \|x\|$;
5′. $\|x^*\| = \|x\|$ **;
6′. $x^* x + e$ possesses a two-sided inverse element for all $x \in R$.

** The authors suppose the last two axioms to be corollaries of 1′—4′, but they have not succeeded in proof of this fact. We also note that the axioms 4′, 5′ may be replaced by the axiom: $\|x^* x\| = \|x\|^2$. For (γ) § 1 implies $\|x\|^2 = \|x^* x\| \leqslant \|x^*\| \cdot \|x\|$, hence

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Settled in 1952

(Fukamiya, Kaplansky)

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The main result

The main purpose of this paper is the proof of the following.

Theorem 1. *Every normed *-ring can be isomorphically mapped onto a closed subring R_1 of the set B of all bounded operators in a Hilbert space $\mathfrak{H}$ in such a manner that, if $x \in R$ and $X \in R_1$ correspond to each other, then $\|x\| = \|X\|$ and x^*, X^* also correspond to each other by this mapping.*

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What else is in the paper?

The Gelfand–Naimark theorem

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What else is in the paper?

Results about von Neumann algebras...

Theorem 2. *Every factor R of class I_∞ or II_∞ in the separable Hilbert space is not simple. It possesses only one* non-trivial *two-sided ideal closed in the uniform topology, which coincides with the smallest two-sided ideal I_0 containing all finite projections and closed in the uniform topology.*

Further development of C^*-algebras, East and West

Work of Gelfand et al., 1939–1950

Commutative C^*-algebras Locally compact abelian groups	Abstract C^*-algebras	Representations of noncompact groups
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Irving Segal (1918–1998)

Further development of C^*-algebras, East and West

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1947 twin papers

The group algebra of a locally compact group

Irreducible representations of operator algebras

The rise of C^*-algebra theory

Irving Kaplansky (1917–2006)

Richard Kadison (1925–2018)

Jacques Dixmier (b. 1924)

Roger Godement (1921–2016)

Further development of C^*-algebras, East and West

Irving Segal (1918–1998)

1947 twin papers

The group algebra of a locally compact group

Irreducible representations of operator algebras

The rise of C^*-algebra theory

Next lecture:

Representation theory for noncompact groups

& loose ends...