RTNCG before 1950, lecture 2 : Locally compact abelian groups in the 1930s

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Previous lecture:

- Peter–Weyl theorem (1927) as a synthesis of **representation theory and spectral theory**
- Connection between **Fourier analysis** and group theory

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Spectral theorem $\subset$ Representation theory

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- Weyl's hints at the representation theory of $\mathbb{R}$:
  - Every unitary rep. of $\mathbb{R}$ reads $t \mapsto e^{itA}$, A unbounded self-adjoint operator in Hilbert space,
  - Connection with the theory of **almost periodic functions** of H. Bohr

Today's subject:

Abelian locally compact groups and harmonic analysis

Mostly between 1933 and 1936

Landmarks for abelian locally compact groups...

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- **Pontryagin 1934**: *The theory of topological commutative groups*
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George Mackey's explanation:

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A strange coincidence...

- **Pontryagin 1933**: *Les fonctions presque périodiques et l'analysis situs*
- **Van Kampen 1936**: *Almost periodic functions and compact groups*
- **Weil 1935**: *Sur les fonctions presque périodiques de von Neumann*

Analysis behind the curtain...

Almost periodic functions on $\mathbb{R}$

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Chevalley and Weil (in an obituary of Weyl):

One can even say of this theory,
which gave rise to so much excitement for about ten years
after Bohr's first publications in 1924,
that its principal merit is to have eased the transition
from the old to the modern point of view about representations...

Harald Bohr (1887–1951)
and his older brother

Almost periodic functions on $\mathbb{R}$

- A function $f : \mathbb{R} \to \mathbb{C}$ is **almost periodic** if it is a uniform limit of trigonometric polynomials.

Finite combinations of functions $x \mapsto e^{i\lambda x}$, $\lambda \in \mathbb{R}$.

Almost periodic functions on $\mathbb{R}$

- A function $f : \mathbb{R} \to \mathbb{C}$ is **almost periodic** if it is a uniform limit of trigonometric polynomials.
- Want to write f as "**Bohr–Fourier series**"

$$f(t) = \sum_{n\in\mathbb{N}} c_{\alpha_n}(f)e^{i\alpha_n t}$$

where the $(\alpha_n)_{n\in\mathbb{N}}$ are real numbers, not necessarily multiples of a ground frequency.

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- Bohr's **Mean value theorem:**

$$\frac{1}{T}\int_0^T f \quad \text{has a limit as } T \to \infty.$$

call this $M[f]\cdots$

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- Bohr's **Fourier coefficients:** For $\lambda \in \mathbb{R}$, set

$$c_\lambda(f) = M\left[f(x)e^{-i\lambda x}\right].$$

Then $\lambda \mapsto c(\lambda)$ has **countable support**.

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- f if almost periodic if and only if
$$\forall \varepsilon > 0,\ \exists \ell > 0 \text{ such that any interval of length } \ell \text{ contains an } \varepsilon\text{-quasiperiod of } f,$$
$$\text{i.e. a number } \tau \text{ such that } \|f(\square + \tau) - f\|_\infty < \varepsilon.$$

Proofs: about 200 pages...

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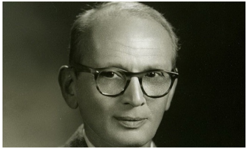
1926–1927: alternate proofs of Bohr's results...

Weyl, Wiener, Bochner, de la Vallée Poussin, Besicovitch...

Bochner and von Neumann's take on almost periodic functions:



Salomon Bochner (1899–1982)

Bochner 1927

$$f \in \mathscr{C}_b(\mathbb{R}) \text{ is almost periodic}$$
$$\Updownarrow$$
$$\boxed{\{f(\square + \tau), \tau \in \mathbb{R}\}} \text{ is } \textbf{relatively compact in } \mathscr{C}_b(\mathbb{R}).$$

Set of translates

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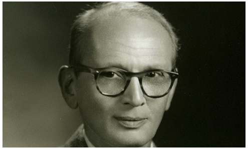

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Von Neumann 1934

- This gives a notion of **a.p. function on any group G**:

  $$f \text{ is right a.p.} \iff \{f(\square\, g), g \in \mathbb{R}\} \text{ is relatively compact in } \mathscr{C}_b(G).$$

- **Peter–Weyl methods** give analogues of Bohr's best theorems

A Paley–Wiener theorem (1932)

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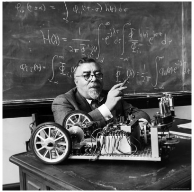
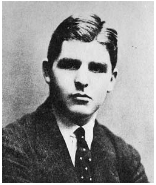


Norbert Wiener (1894–1964)



Raymond Paley (1906–1933)

Wiener 1932

TAUBERIAN THEOREMS.*

BY NORBERT WIENER.

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LEMMA IIe. *If $f(x)$ is a function with an absolutely convergent Fourier series, which nowhere vanishes for real arguments, $1/f(x)$ has an absolutely convergent Fourier series.*

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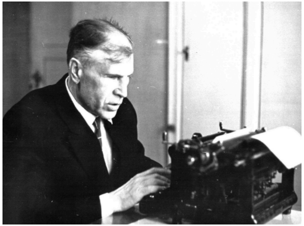
- The above theorem **extends to Bohr–Fourier series** of almost periodic functions on $\mathbb{R}$
- G: abelian group, assume its character group has an invariant measure
- Use Bochner and von Neumann's definition of **almost periodic functions on groups**
  $\rightarrow$ analogue of Wiener's theorem

Back to the main road...

Enter Pontryagin



Lev Pontryagin (1908–1988)

ANNALS OF MATHEMATICS
Vol. 35, No. 2, April, 1934

THE THEORY OF TOPOLOGICAL COMMUTATIVE GROUPS

BY L. PONTRJAGIN[1]

(Received November 22, 1933; Revised March 6, 1934)

Introduction

The purpose of the present paper is to make an exhaustive investigation into the structure of continuous, locally compact, commutative groups, satisfying the second axiom of countability.[2]

An important motivation: Alexander duality in topology

- Suppose $K \subset \mathbb{R}^n$ compact subset.
- Want to study relationship between $H_i(K)$ and $H_{n-i}(\mathbb{R}^n - K)$.
- Way to go: prove one is the character group of the other.

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- **Stepanoff–Tychonoff:** if f is almost-periodic, then
  - Consider **closure** $R(f) \subset \mathscr{C}_b(G)$ **of the translates of** f
  - Can equip $R(f)$ with group structure $\rightarrow$ **compact abelian group with** $\mathbb{R}$ **dense subgroup.**

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  - Can equip $R(f)$ with group structure $\to$ **compact abelian group with** $\mathbb{R}$ **dense subgroup.**
- **Pontryagin:** let $\alpha_1, \ldots, \alpha_n, \ldots$ be the frequencies of f, then
  - $R(f) \simeq \prod_i \mathbb{R}/(\alpha_i \mathbb{Z})$
  - This is the **dual group of** $\bigoplus_i \mathbb{Z}\alpha_i$**.**

What's in Pontryagin's paper?

First Fundamental Theorem. *Let Ω be a continuous compact commutative group satisfying the second axiom of countability, and $\mathfrak{G}$ the discrete group of its characters, then the group Ω is isomorphic to the group of characters of $\mathfrak{G}$.* (See Definitions 1 and 1′.)

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The Second Fundamental Theorem: *A locally compact connected group Ω satisfying the second axiom of countability decomposes into a direct sum of a compact subgroup Δ, and a vector subgroup* N (*see Definition* 4), *where the subgroup Δ is determined uniquely, and the demensionality r of the group* N *is an invariant of the group Ω.*

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Third Fundamental Theorem. *A connected, locally connected, locally compact group Ω, satisfying the second axiom of countability, decomposes into the direct sum of a finite or denumerable number of continuous cyclic groups and a vector group.*

Digression: why locally compact groups?

- **Haar 1933**: existence of an invariant measure for **second countable locally compact groups**

Digression: why locally compact groups?

In der vorliegenden Arbeit soll gezeigt werden, daß bei jeder N-gliedrigen kontinuierlichen Gruppe ein solcher Inhalts- bzw. Maßbegriff tatsächlich vorhanden ist. Unsere Untersuchungen gelten sogar für noch allgemeinere kontinuierliche Gruppen; wir werden im wesentlichen nur annehmen, daß die *Gruppenmannigfaltigkeit metrisch, separabel und im Kleinen kompakt ist.* § 1 enthält die Konstruktion des Inhaltsbegriffes; im § 2 wird gezeigt,

1

2

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What Haar says:

1 We'll show the existence of an invariant measure if G is a topological manifold

2 In fact we shall only need to assume G is metric, separable, locally compact...

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Hilbert's 5th problem:

If G is a topological manifold (and multiplication is continuous), then is G a Lie group?

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- **Haar 1933**: existence of an invariant measure for **second countable locally compact groups**
- Used by **von Neumann (1933):** solution of **Hilbert's fifth problem for compact groups**
- Extension to **all locally compact groups** in Weil's book (see below)

Van Kampen's work

Egbert van Kampen (1908–1941)

Von Neumann remarked that the whole theory can be extended without any additional effort to bicompact and (unrestricted) discrete groups.

A modification of Pontryagin's result could be extended [...], according to von Neumann's remark mentioned above, to all locally compact Abelian groups.

Van Kampen's work

Cf. Pontryagin

THEOREM 1. *A one-to-one correspondence can be established between all B-groups and all D-groups in such a way that each group is the character group of the corresponding group.*

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3. THEOREM 2.

Classification result

a. An A-group G can be brought into the form $[T + R \frown S]$. Here T is a translation group, R is a B-group, S is a D-group.

b. T can be any maximal translation subgroup of G independent of the particular group $R \frown S$ chosen among the possible ones.

c. $R \frown S$ contains the sum F of all bicompact subgroups of G. The factor group G/F has a translation subgroup as (isolated) component of zero and is the direct sum of that component and any subgroup E of G/F, meeting each component in one point. For $R \frown S$ we can take the subgroup of G generated by F and any such group E.

(For a modern version see Bourbaki, *Théories spectrales*, Chap. 2)

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Classification result

(For a modern version see Bourbaki, *Théories spectrales*, Chap. 2)

THEOREM 3. *a. Among all A-groups a one-to-one correspondence can be established such that each A-group is the character group of its corresponding group.*

b. If an A-group is brought on its normal form $[T + R \frown S]$, its character group has the form $[\bar{T} + \bar{S} \frown \bar{R}]$, where $\bar{R}$, $\bar{S}$, $\bar{T}$ are the character groups of R, S, T.

Application to almost periodic functions

Bohr compactification

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  $\rightarrow$ this is a **compact abelian group**, and G embeds canonically into $\widetilde{G}$.

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Going out with a blaze

- Restriction to G of continuous functions on $\widetilde{G}$ = almost periodic functions on G
- **Peter–Weyl theory for $\mathscr{C}(\widetilde{G})$ $\rightarrow$ all known results on almost periodic functions**!

Positive-definite functions

Notion of positive-definite function

- **O. Toeplitz (1911)** defines **sequence $(u_n)_{n\in\mathbb{Z}}$ of complex numbers** to be positive-definite when

  Hermitian matrix $(c_{i-j})_{1\leqslant i,j\leqslant n}$ is non-negative for all n.

- Studied intensively in the 1910s:

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- **M. Mathias (1923)** defines $f : \mathbb{R} \to \mathbb{C}$ **to be positive-definite** when

  Hermitian matrix $(f(x_i - x_j))_{1\leqslant i,j\leqslant n}$ is non-negative for all n and all $x_1, \ldots, x_n \in \mathbb{R}$.

- **Bochner (1932)** makes them a central tool in harmonic analysis and probability

Positive-definite functions

Positive-definite functions

Bochner's theorem

$$\{ \text{ Positive-definite functions on } \mathbb{R} \text{ } \}$$

$$\updownarrow \text{ Fourier transform}$$

$$\{ \text{ Positive measures on } \mathbb{R} \text{ of finite total mass } \}$$

Riesz 1933 (also Bochner, independently):

- If $\pi : \mathbb{R} \to \mathrm{End}(\mathcal{H})$ unitary rep., then

$$c_f(t) := t \mapsto \langle f, \pi(t) f \rangle \text{ is positive-definite for all } f \in \mathcal{H}$$

- Leads to a simple proof of Stone's theorem

André Weil's synthesis

André Weil (1906–1994)

1936 book (published in 1940)

- Contains **simplified expositions** of
  - Haar measure

Including more generality than Haar...

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  - Peter–Weyl theorem
  - Duality for locally compact abelian groups

André Weil (1906–1994)

1936 book (published in 1940)

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  - Haar measure
  - Peter–Weyl theorem
  - Duality for locally compact abelian groups
- Emphasis on **integration and harmonic analysis**
  - Systematic use of **convolution product**
  - General discussion of **positive-definite functions**

New results in Weil's book

- **Introduces Fourier transform** in abelian case
- Proves **Plancherel formula and Bochner theorem**
- Detailed discussion of **almost periodic functions**

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- Proofs **still rely on classification results** for structure of locally compact abelian groups

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  This used the methods introduced by Gelfand's school in the early 1940s...

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- Very influential book... in the West!

Next lecture:

The birth of C^*-algebras

& Gelfand's school in the 1940s