RTNCG before 1950, lecture 1 : Compact groups in the 1920s

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- RT: representation theory of locally compact groups,
- NCG: structure of C^*-algebras, connections with representation theory
  (+ index theory, K-theory...)

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- *Sur les relations d'orthogonalité de Bargmann*, by R. Godement.

Aim of the lectures:

- Short stories from 1927–1947, to understand the simultaneous birth of the two subjects...
- ... told from a naive perspective, neither a serious historian's nor an experienced mathematician's.

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For references :

http://afgoustidis.perso.math.cnrs.fr/RTNCG_before_1950.html

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Mini-mini-mini:

- Informal, short lectures
- Small lies are allowed (change of notation, anachronistic shortcuts...)

Today's real subject:

The Peter–Weyl paper (1927)

which seems to be the first bridge between representation theory and analysis.

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The Peter–Weyl paper (1927)

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- Hermann Weyl's work and background, 1904 – 1924
- The Peter-Weyl paper itself

Some exciting developements from the period 1900–1924

Analysis : Spectral theory

- Hilbert's work on integral equations, Hilbert–Schmidt spectral theory (1900–1910),
- F. Riesz, compact operators (1916)

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- Representations of nonabelian finite groups, ca. 1900–1905 (Frobenius, Burnside, Schur...)

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Algebra : finite groups

- Representations of nonabelian finite groups, ca. 1900–1905 (Frobenius, Burnside, Schur...)

Physics : relativity

- Special and general Relativity
- Mathematical investigations: differential geometry, tensor calculus

Hermann Weyl's early career

- **1904–1914 :** work on integral equations and spectral theory

  (+ Riemann surfaces)

- **1914–1924 :** turns to relativity, and this leads him to:
  - differential geometry,
  - tensor calculus,
  - then finite groups and invariant theory.

- **1925–1926 :** compact groups

Hermann Weyl's early career

Hermann Weyl (1887–1955)

&

David Hilbert (1862–1943)

in Göttingen, ca. 1925

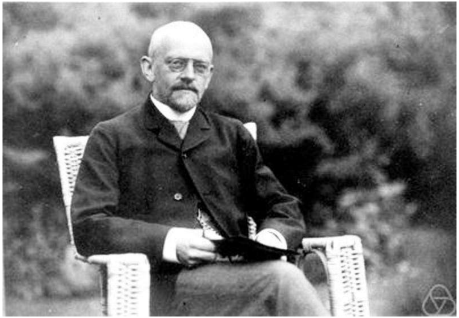
I came to Göttingen a country lad of eighteen.

In the fullness of my ignorance, I made bold to take the course Hilbert had announced...

Most of it went straight over my head. But the doors of a new world swung open for me;

and I had not sat long at Hilbert's feet before the resolution formed itself in my young heart

that I must by all means read and study whatever this man had written.



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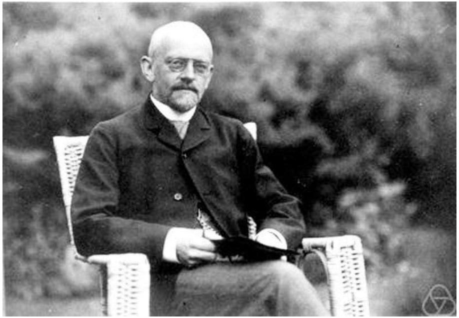

Hilbert's interests:

- 1885-1893: invariant theory
- 1893-1898: algebraic number theory
- 1898-1900: foundations of geometry
- 1900-1912: integral equations

Hilbert, integral equations and the birth of spectral theory

Hilbert, integral equations and the birth of spectral theory

- Around 1900, Fredholm had studied the equation

$$u(x) - \lambda \int_a^b K(x,y)u(y)dy = f(x);$$

(Solve for u , given continuous f on [a,b] and continuous kernel K on [a,b]x[a,b])

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$(1 - \lambda A_K) u = f$ where $A_K : \mathscr{C}([a,b]) \longrightarrow \mathscr{C}([a,b])$

$u \longmapsto \left(x \mapsto \int_a^b K(x,y)\, u(y)\, dy \right)$

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- Parameter $\lambda \longrightarrow$ solution as meromorphic function of λ.
  but Fredholm doesn't view this as eigenvalue pb.

Hilbert 1904:

$$u(x) - \lambda \int_a^b K(x,y)u(y)dy = 0$$

- if K is **symmetric**, then analogy with **quadratic forms** and principal axes theorem.

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$$\frac{1}{\text{eigenvalues of } A_K \text{ in modern sense}}$$

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- if K is **symmetric**, then analogy with **quadratic forms** and principal axes theorem.
- Countably many values $\lambda_1, \lambda_2, \ldots$ for which there can be a nontrivial solution ("eigenvalues").
- If $\varphi_1, \varphi_2, \ldots$ is a corresponding orthonormal family of "eigenfunctions", then

$$\int_a^b \int_a^b K(s,t)f(s)g(t) = \sum_k \frac{1}{\lambda_k} c_k(f)c_k(g)$$

where $c_k(f) = \int_a^b f(s)\varphi_k(s)ds$ is viewed as a "Fourier coefficient" of f.

Hilbert 1906:

- More generally, can consider **quadratic forms on $\ell^2(\mathbb{N})$**,

$$B(x) = \sum_{p,q} b_{p,q}\, x_p\, x_q$$

$x = (x_1, x_2, \dots) \in \ell^2(\mathbb{N})$

and try to find "reduction"

$$B(x) = \sum_k \lambda_k (x'_k)^2.$$

x'_k : coordinates in another basis of $\ell^2(\mathbb{N})$.

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If $x \to x_*$ **weakly**, then $B(x) \to B(x_*)$.

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- But can consider arbitrary **"bounded"** forms... and Hilbert discovers **continuous spectrum**.
- Very many applications to "singular" Fredholm-type equations, and other problems.
- *In the terrain of analysis a rich vein of gold has been struck...*

Work of E. Schmidt on Hilbert's results

1905 thesis

Erhard Schmidt
(1876–1959)

- Elegant, constructive approach to eigenvalues and eigenfunctions.

...without infinite determinants...

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$$(x, y) \mapsto \int_a^b K(x, z)K(y, z)dz.$$

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Corresponding endomorphism of $C([a,b])$ is $A_K A_K^*$

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1908

- Geometric perspective on $\ell^2(\mathbb{Z})$ and Hilbert's results... that's another story.

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Big story
(functional analysis)

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Further contributions by F. Riesz

1913 book:

- Instead of bilinear form $B(x,y)$ on $\ell^2(\mathbb{Z})$, consider **endomorphism** A s.t.

$$B(x,y) = \langle x, Ay\rangle_{\ell^2(\mathbb{Z})}.$$

- Modern notion of spectrum, functional calculus...
- **Completely continuous** forms become **compact operators** on $\ell^2(\mathbb{Z})$.

1916 paper:

- General **spectral theory of compact operators** (ostensibly on $\mathscr{C}([a,b])$).

Frygies Riesz
(1880–1956)

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  - then finite groups and invariant theory.

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Weyl, general relativity and tensor calculus after 1916

My mathematical mind was as blank as any veteran's,

and I did not know what to do.

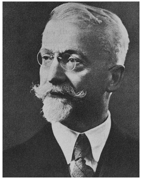
I started to study algebraic surfaces;

but before long, Einstein's memoir came into my hand and set me afire.

From relativity to group theory

New topics for Weyl...

Élie Cartan (1869–1951)



- Differential geometry, work of Élie Cartan
- Lie algebras
- Tensor calculus

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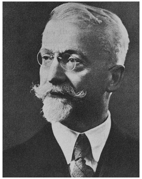

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Controversy with E. Study

Eduard Study (1862–1930)

- Study complained about the neglect of invariant theory by Weyl and others.
- Led Weyl to several papers on invariant theory and finite groups
- End of 1924: complete reductibility of finite-dim. reps. of $\mathrm{SL}(n, \mathbb{C})$

From relativity to group theory

Weyl's annual report, January 1924

The theory of finite groups, to which I was originally drawn by the theory of relativity, is becoming more and more my real area of work.

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Weyl's annual report, January 1925

In the last few months I succeeded in overcoming the difficulties, based in part on researches of Prof. É. Cartan, Paris, and partly in collaboration with Prof. Schur, Berlin. The results combine into one of the most wonderful theories to be found in mathematics.

Flashback to Frobenius and Schur, 1896–1905

Georg Frobenius (1849–1917)

Issai Schur (1875–1941)

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Georg Frobenius (1849–1917)

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Frobenius 1896–1900

- Definitions and basic notions for representations of finite groups
- **Complete reducibility** for finite-dimensional representations
- **Regular representation theorem**: contains all irreducibles, with multiplicity the degree

  → Implicit in proof: idempotents in the group algebra

Schur 1901–1905

- **1901 thesis:** invariant theory and **polynomial representations of** $\mathrm{GL}(n, \mathbb{C})$

  $\rightarrow$ connection with representations of $\mathfrak{S}_n$

- **1905:** reworking of the Frobenius theory
  - Emphasis on elementary methods, Schur's lemma as a key tool
  - **Orthogonality relations for matrix coefficients**

Hurwitz 1897

- Had used "invariant integration" to study the invariants of $\mathrm{SO}(n, \mathbb{R})$.

Schur's 1923–1924 inspiration, and his correspondence with Weyl

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- Had used "invariant integration" to study the invariants of $\mathrm{SO}(n, \mathbb{R})$.

Schur 1923–1924

- Realized that Hurwitz's method carried over his orthogonality relations to compact groups.
- Published "New applications of the Integral calculus to problems in the theory of invariants".
- In the next months, for irreducible represenations of $\mathrm{SO}(n, \mathbb{R})$ and $\mathrm{O}(n, \mathbb{R})$, found
  - Character formula
  - Dimension formula

Schur's 1923–1924 inspiration, and his correspondence with Weyl

- Weyl sent a letter to Schur in October 1924, with
  - an offprint of his recent "invariant theory" paper,
  - a draft of his complete reducibility theorem for $\mathrm{SL}(n, \mathbb{C})$,
    $\rightarrow$ using the "unitary trick".

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    $\rightarrow$ using the "unitary trick".
- Schur replied with a description of his character and dimension formulas, comparing them to work of Cartan.
- Less than two weeks later, Weyl had the results for *all semisimple compact Lie groups.*

Weyl's 1925 papers on compact semisimple Lie groups

- **Complete reductibility** of finite-dimensional representations of complex semisimple Lie algebras

- Compact form, compact **covering group**, unitary trick

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- Structure theory beyond Cartan–Killing: root **reflections, Weyl group**...
- **Character and dimension formulas**, modelled on Schur's.

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- Structure theory beyond Cartan–Killing: root **reflections, Weyl group**...
- **Character and dimension formulas**, modelled on Schur's.
- **Left open:** analogue of the "regular representation theorem" of Frobenius?

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Who was Fritz Peter?

The Peter–Weyl paper

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(Use at your own risk!)

- Quite short (19 pages)
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Peter–Weyl preliminaries

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Alfred Haar 1933! Motivated ... by Peter-Weyl.

Peter-Weyl assume G is a compact Lie group to have invariant measure ...

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- If π is a continuous representation of G on a finite-dim. vector space V_π, then π is unitarizable.

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Matrix coefficients

- Fix an orthonormal basis $(v_1, \ldots, v_{n_\pi})$ of V_π, and define $c_{ij}^\pi : G \to \mathbb{C}$ by

$$c_{ij}^\pi(g) := \langle v_i, \pi(g) v_j \rangle.$$

- Schur relations:
  - $\left\| c_{ij}^\pi \right\|_{\mathrm{L}^2(G)}^2 = \frac{\mathrm{Vol}(G)}{n_\pi}$,
  - and if $\pi \not\simeq \pi'$, then $\langle c_{ij}^\pi, c_{kl}^{\pi'} \rangle_{\mathrm{L}^2(G)} = 0$ for all i, j, k, l.

The strategy

Peter & Weyl's statement of the result

- Schur's relations mean the collections $(n^{1/2}\mathrm{Vol}(G)^{-1/2}c_{ij}^{\pi})$, for the various inequivalent π, give rise to an **orthonormal family** in $\mathrm{L}^2(G)$.
- Peter–Weyl: we'll prove that **this orthonormal system is complete**.

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Enter ideas from Fourier analysis

- For $f : G \to \mathbb{C}$ continuous, and π as above, form the "Fourier coefficients"
$$\pi(f) = \int_G \overline{f(s)}\pi(s)ds, \qquad c^{\pi}_{ij}(f) = \langle v_i, \pi(f)v_j\rangle$$
- Then Schur's relations immediately give "Bessel inequality"
$$\sum_{\pi}\sum_{i,j} n_{\pi}|c^{\pi}_{ij}(f)|^2 \leqslant \mathrm{Vol}(G)\cdot\|f\|^2 .$$

On page 5, C^*-algebras!

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- For continuous functions on G, can consider
  - **convolution product** $f * g = \int_G f(xy^{-1})g(y)dy$;
  - **hermitian conjugate** $f \mapsto \widetilde{f}$, where $\widetilde{f}(g) = \overline{f(g^{-1})}$,
  - **trace** of an element: $\mathrm{Tr}(f) = \mathrm{Vol}(G) \cdot f(1_G)$.
- Can rewrite Bessel inequality as

$$\sum_\pi n_\pi \,\mathrm{Tr}\left(\pi(f)\pi(f)^\star\right) \leqslant \mathrm{Tr}(f * \widetilde{f}).$$

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- Can rewrite Bessel inequality as
$$\sum_\pi n_\pi \,\mathrm{Tr}\left(\pi(f)\pi(f)^\star\right) \leqslant \mathrm{Tr}(f * \widetilde{f}).$$
- Notice **product rule** $\pi(f * g) = \pi(f)\pi(g)$, and $\pi(\widetilde{f}) = \pi(f)^\dagger$. Thus can rewrite as
$$\sum_\pi n_\pi \,\mathrm{Tr}\,\pi(f * \widetilde{f}) \leqslant \mathrm{Tr}(f * \widetilde{f}).$$
- Idea: notice all operators $\pi(f * \widetilde{f})$ are **hermitian positive**, and apply Hilbert–Schmidt theory.

The Peter–Weyl argument, in outline

$$\boxed{\sum_{\pi} n_\pi \operatorname{Tr} \pi(f * \widetilde{f}) \leqslant \operatorname{Tr}(f * \widetilde{f}).}$$

- Instead of the operators

$$\pi(f * \widetilde{f}) \in \operatorname{End}(V_\pi) \quad \text{for the various possible } \pi,$$

consider $z = f * \widetilde{f}$ and the kernel

$$\begin{aligned} K_z : G \times G &\to \mathbb{C} \\ (x, y) &\mapsto (f * \widetilde{f})(xy^{-1}). \end{aligned}$$

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$$\begin{aligned} K_z : G \times G &\to \mathbb{C} \\ (x, y) &\mapsto (f * \widetilde{f})(xy^{-1}). \end{aligned}$$

- If $\lambda_1, \ldots, \lambda_n$ eigenvalues of $\pi(f * \widetilde{f})$, and $v_1, \ldots, v_n \in V_\pi$ basis of eigenvectors, then

  For fixed i, **matrix coefficients** $c_{i1}, \ldots, c_{in}$ **are eigenfunctions** of K_z for $\gamma_1, \ldots, \gamma_n$.

- Will **reduce theorem** to the fact that $\operatorname{Tr}(z) = \operatorname{Tr}(f * \widetilde{f})$ is the **sum of the eigenvalues** of K_z.

The Peter–Weyl argument, in outline

Key step: **construct representations** π **such that** $\pi(z) = \pi(f * \widetilde{f})$ **is nonzero**.

- The endomorphism A_z of $\mathrm{L}^2(G)$ is compact and positive.

$\rightarrow$ Nonzero eigenvalues have **finite multiplicities**, and can be arranged in sequence

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- Can then find an orthonormal family $\varphi_1, \dots, \varphi_n$ in the λ_1-eigenspace for K_z such that

$$e(st^{-1}) = \varphi_1(s)\overline{\varphi_1(t)} + \dots + \varphi_n(s)\overline{\varphi_n(t)}.$$

- Functions $\varphi_1, \dots, \varphi_n$ **span a finite-dimensional subrepresentation** π of $L^2(G)$, and $\pi(z) \neq 0$.

The Peter–Weyl argument, in outline

Done so far:

- Beginning with arbitrary nonzero f, and setting $z = f * \widetilde{f}$, have built π such that $\pi(z) \neq 0$.
- Can assume π irreducible, form matrix coefficients c_{ij}^{π} and **finite Fourier series**

$$S_f^{(1)} = \sum_{i,j} c_{ij}^{\pi}(f) c_{ij}^{\pi}$$

- Can consider **remainder** $f^{(1)} = f - S_f^{(1)} = \sum_{i,j} c_{ij}^{\pi}(f) c_{ij}^{\pi}$.

Iterate previous arguments:

- Get "partial sums" – finite Fourier series $S_f^{(p)}$ comprising more representations, and **remainders** $(f^{(p)})$ at each step.
- Explicit estimate proves **remainder** $f^{(p)}$ **goes to** 0, both uniformly and in L^2, as $p \to \infty$.

Peter and Weyl's concluding remarks

1. "Fourier expansion" converges **uniformly**, not only in L^2 sense.

2. **All irreducible representations must occur** in regular representation.

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   Proof:

   - Would be easy if convolution algebra $\mathscr{C}(G)$ had a unit. But it doesn't.
   - Instead consider **approximate unit** $(1_\nu)_{\nu\in\mathbb{N}}$...
   - Then for all π, must have $\pi(1_\nu) \to 1$, and therefore $\pi(1_\nu)$ must be nonzero for large ν.

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4. Weyl (1926) had applied similar methods for H. Bohr's theory of **almost periodic functions**.

   $\rightsquigarrow$ *"first example of character theory for a truly noncompact group, that of translations of* $\mathbb{R}$*"*.

Weyl on Hilbert's spectral theory

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But then a sort of miracle happened: the spectral theory in Hilbert space

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Quantum developments to above themes

- **von Neumann 1927**: abstract Hilbert space, unbounded (self-adjoint) operators, spectral theorem
- **Weyl 1928**: sensational book *Gruppentheorie und Quantenmechanik*
- In the book: a conjecture on unitary representations of $\mathbb{R}$

- **Weyl 1928:** if A is a **bounded** **self-adjoint** operator on a Hilbert space H, then

$$t \mapsto e^{itA} \quad \text{is a unitary representation of } \mathbb{R}.$$

- **Weyl 1928:** allow A to be **unbounded**, then maybe **all unitary reps. of** $\mathbb{R}$ arise in this way.

- **Stone 1930:** that is true.

Next lecture:

abelian locally compact groups (1932–1936),

with an emphasis on the analytic background.