On Bismut's Hypoelliptic Laplacian

I shall talk about some work of J.-M. Bismut that is perhaps still a few years ahead of its time.

It has a direct bearing on topics presumably of interest to representation theorists ... e.g. it provides a new formula for

$$\boxed{\int_{\widehat{G}} \operatorname{mult}(\sigma, \pi)\, e^{-t\|\text{inf.ch.}(\pi)\|^2}\, d\mu(\pi)}$$

$\sigma \in \widehat{K}$

Plancherel measure

Bismut's derivation of his formula is wholely derived from (his approach to) index theory.

What Does the Hypoelliptic Laplacian Do, Exactly?

From the point of view of representation theory, the major application is a formula for semisimple orbital integrals. $\left(\int_{\text{conj class}} f(g)\, dg \right)$

This includes non-regular orbits like $\{e\} \subseteq G$.

What kind of formula?

The formula involves the integral heat kernel of

$$\Delta : L^2(G/K, \underline{V}) \longrightarrow L^2(G/K, \underline{V})$$

Laplacian

Vector bundle assoc. to a rep. of K

For instance for $\{e\}$ and for V = triv. rep. of K,

Integral kernel of $e^{-t\Delta}$, evaluated at $(eK, eK) \in G/K \times G/K$

$$\boxed{e^{-t\Delta}(eK, eK) = \frac{e^{-|\rho|^2 t/4}}{(4\pi t)^{\dim(\mathfrak{g})/2}} \int_{\mathfrak{k}} J(X)\, e^{-\|X\|^2/4\pi t}\, dX}$$

$$J(X) = \frac{\hat{A}(i \cdot \mathrm{ad}_X : \mathfrak{s} \longrightarrow \mathfrak{s})}{\hat{A}(i \cdot \mathrm{ad}_X : \mathfrak{k} \longrightarrow \mathfrak{k})}$$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$

$$\hat{A}(T) = \det{}^{1/2}\left(\frac{T/2}{\sinh(T/2)} \right)$$

Harish-Chandra also has a formula... in the same special case it is

$$e^{-t\Delta}(eK, eK) = \int_{\mathfrak{a}^*} e^{-t|\lambda|^2} \frac{d\lambda}{|c(\lambda)|^2},$$

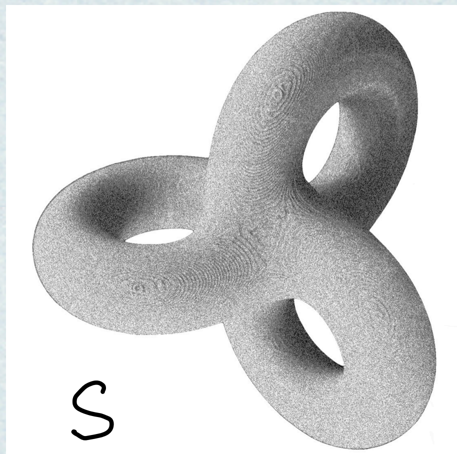
Plancherel measure for spherical principal series

which is different (but equivalent, of course).

Bismut's formula for a general orbital integral (of the function

$$g \longmapsto \exp(-t\Delta)(gK, eK)$$

still) is more complicated but in the same vein.



And for instance he recovers Selberg's formula

$$\sum_{\mathrm{Spec}(\Delta_S)} e^{-t\lambda} = \frac{\mathrm{Area}(S)}{4\pi t} \frac{e^{-t/4}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \frac{x/2}{\sinh(x/2)} e^{-x^2/4t}\, dt$$

$$+ \frac{e^{-t/4}}{\sqrt{4\pi t}} \sum_{\substack{\text{closed} \\ \text{geodesics}}} \frac{\ell_0(\gamma)/2}{\sinh(\ell_0(\gamma)/2)} e^{-\ell(\gamma)^2/4t}$$

complete accuracy is not guaranteed!

What is the Hypoelliptic Laplacian?

From now on I shall consider an extremely simple example — the circle $\mathbb{T}$. (Bismut studied compact groups before studying G/K, and one could study compact symmetric spaces too.)

Ingredients for the hypoelliptic Laplacian:

- $$D = \begin{bmatrix} 0 & \partial/\partial x \\ \partial/\partial x & 0 \end{bmatrix} \qquad (x \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z})$$

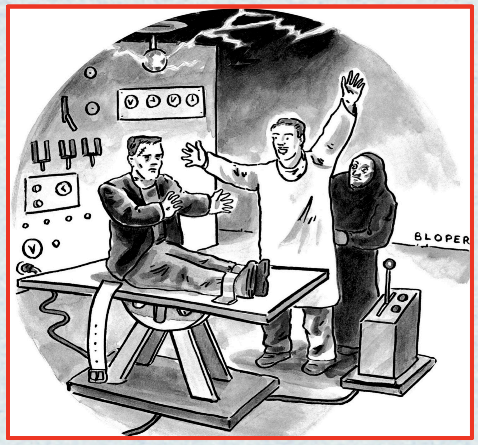
  (More generally Kostant's Dirac op. on G)

- $$Q = \begin{bmatrix} 0 & -\partial/\partial y + y \\ \partial/\partial y + y & 0 \end{bmatrix} \qquad (y \in \mathbb{R})$$

  (More generally Witten's $e^{-f} d e^{f} + e^{-f} d^{*} e^{f}$ on $\mathfrak{g}$)

$$Q^2 = \begin{bmatrix} -\partial^2/\partial y^2 + y^2 - 1 & 0 \\ 0 & -\partial^2/\partial y^2 + y^2 + 1 \end{bmatrix}$$

The ingredients are rather ordinary, but they are combined in a decidedly unorthodox fashion...



<u>Definition</u> "The" hypoelliptic Laplacian on the circle is the family of operators

$$L_b = \frac{1}{2}\left(\frac{1}{b}Q + D\right)^2 - \frac{1}{2}D^2$$

on $\mathbb{T} \times \mathbb{R}$ (generally, on $G \times \mathfrak{g}$) parametrized by $b > 0$.

$$\boxed{L_b = \begin{bmatrix} \frac{1}{2b^2}\left(-\partial^2/\partial y^2 + y^2 - 1\right) + \frac{1}{b} y \partial/\partial x & 0 \\ 0 & \frac{1}{2b^2}\left(-\partial^2/\partial y^2 + y^2 + 1\right) + \frac{1}{b} y \partial/\partial x \end{bmatrix}}$$

This is

- Not a Laplacian
- Not positive-definite
- Not even self-adjoint

(However it is indeed hypoelliptic.)

b-Independence of the Supertrace

Despite its demerits, the hypoelliptic Laplacian has a number of remarkable properties...

In the first lecture, we noted (in the context of Lefschetz theory) that

$$\sum (-1)^p \operatorname{Trace}\left(e^{-t\Delta} : \Omega^p(M) \longrightarrow \Omega^p(M)\right)$$

is independent of $t > 0$.

Something similar is true here:

Theorem The quantity

$$\operatorname{STr}\left(e^{-tL_b}\right) = \operatorname{Tr}\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} e^{-tL_b}\right)$$

is independent of $b > 0$ (but not $t > 0$).

This is a simple algebraic fact (like the chain homotopy argument for Lefschetz) based on the fact that D^2 commutes with Q (and this explains the use of Kostant's D, in general).

Not so simple analytic fact: the traces make sense in the first place! (Kolmogorov, Hormander, ...)

Large b-Limit

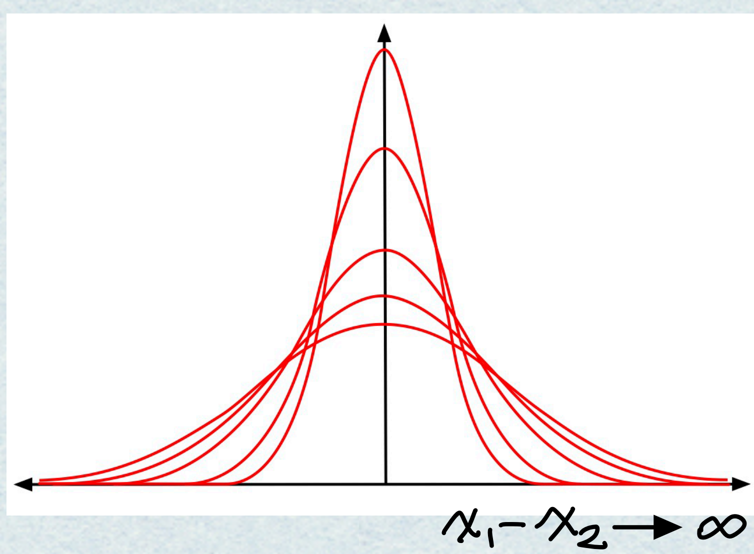
General plan: get a formula by studying $b \longrightarrow \infty$ (this is like studying $t \longrightarrow 0$ in the Lefschetz story) and by studying $b \longrightarrow 0$ (this step doesn't really exist in the Lefschetz story — or at least it is very easy; here it is not).

Remark As for t, in the Bismut story it will remain fixed.

Carrying out the plan is technically difficult because L_b is not a Laplacian, not elliptic, not self-adjoint, etc.

One also needs to exploit two features of L_b that were brilliantly engineered into the definition by Bismut...

Here is a picture of $\exp(-t\Delta)(x_1, x_2)$ for $M = \mathbb{T}$ and various values of $t > 0$.



The "concentration property" was exploited in Lecture 1.

And here are pictures of the functions

$$w \longmapsto \mathrm{str}\Big(\exp(-tL_b)\big((0,bw),(0,bw)\big)\Big)$$

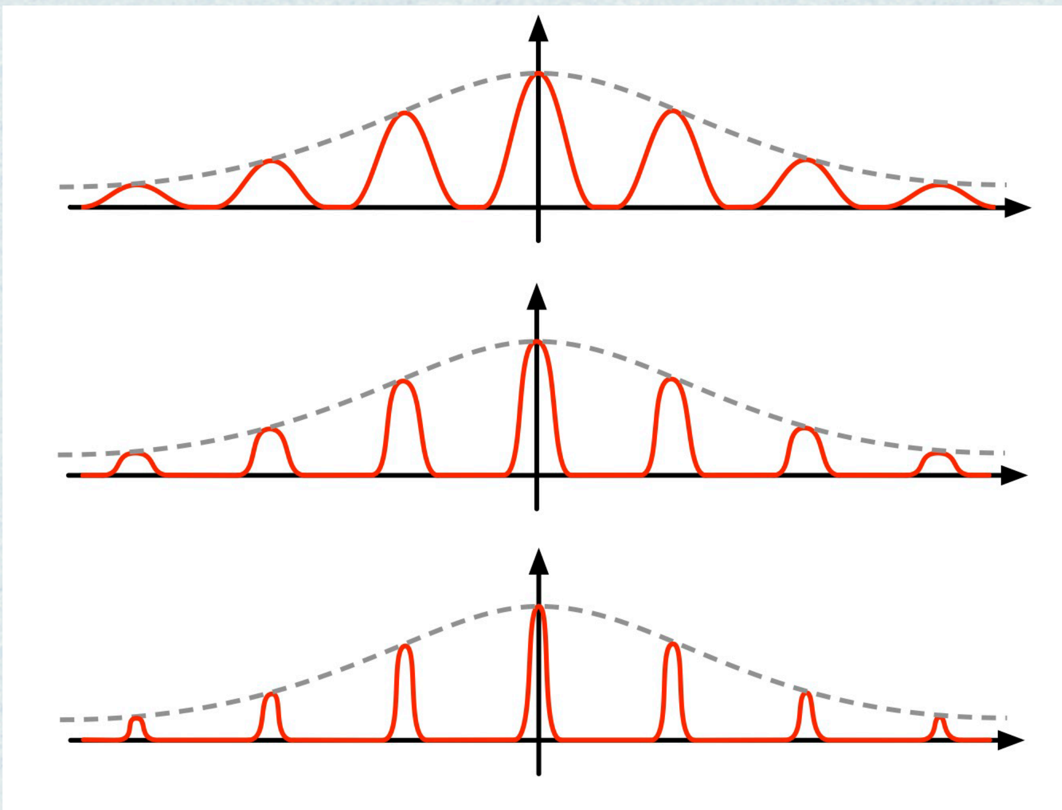
(multiply by b and integrate over y to get b-independent operator supertrace)

for various values of $b>0$, with $b \to \infty$ going downwards.

b

$\downarrow$

∞



Graphs of $w \longmapsto \mathrm{str}\Big(\exp(-tL_b)\big((0,bw),(0,bw)\big)\Big)$

The "concentrations" occur at integers $\cdot\, {}^{1}/_{t}$ and they arise from the term $y\,{}^{\partial}/_{\partial x}$ in L_b. Note that this generates the geodesic flow on $\mathbb{T}\times\mathbb{R} \cong$ tangent bundle of the circle.

Shift Property

Let $\Delta_{\mathbb{T}\times\mathbb{R}} = -\partial^2/\partial x^2 - \partial^2/\partial y^2$. Solutions to

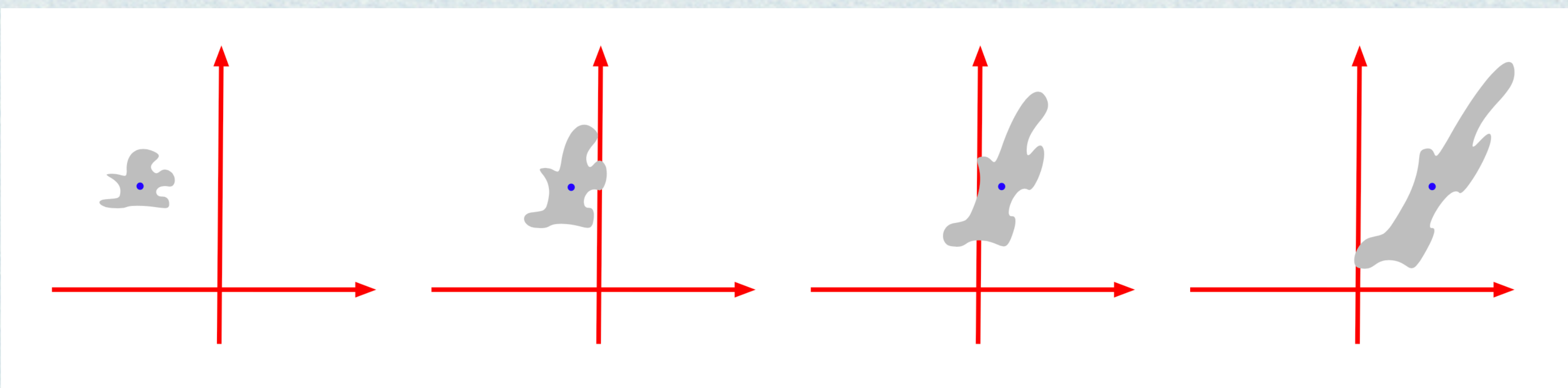
$$\frac{\partial u_t}{\partial t} = -\Delta_{\mathbb{T}\times\mathbb{R}}\, u_t$$

diffuse in the x- and y-directions.

In contrast, if $K = -\partial^2/\partial y^2 + y\,\partial/\partial x$ (an operator first studied by Kolmogorov) then solutions to

$$\frac{\partial u}{\partial t} = -Ku$$

diffuse in the y-direction, but drift in the x-direction at a rate proportional to y.



It follows that for large b the integral kernel

$$\exp(-tL_b)\big((x_1,y_1),(x_2,y_2)\big)$$

concentrates not on the diagonal

$$\Delta(\mathbb{T}\times\mathbb{R}) \subseteq (\mathbb{T}\times\mathbb{R})\times(\mathbb{T}\times\mathbb{R})$$

but on the shifted diagonal

$$\left\{(x_1,y_1) = \left(x_2+\tfrac{t}{b}y_2,\ y_2\right)\right\}$$

And note that

$$\boxed{\begin{aligned}&(\text{Shifted Diagonal}) \cap (\text{Standard Diagonal})\\ &\qquad = \left\{\left(\left(x,\tfrac{b}{t}n\right),\left(x,\tfrac{b}{t}n\right)\right) : n\in\mathbb{Z}\right\}\end{aligned}}$$

For $w = by$, the intersection is $\left\{w\in \frac{1}{t}\mathbb{Z}\right\}$

Small b-Limit

Remember that

$$L_b = \left(\tfrac{1}{b}Q + D\right)^2 - D^2$$

$$= \begin{bmatrix} \frac{1}{2b^2}\left(-\partial^2/\partial y^2 + y^2 - 1\right) + \frac{1}{b}y\,\partial/\partial x & 0 \\ 0 & \frac{1}{2b^2}\left(-\partial^2/\partial y^2 + y^2 + 1\right) + \frac{1}{b}y\,\partial/\partial x \end{bmatrix}$$

It is a remarkable fact that despite having "subtracted out" D^2 the operator $\partial^2/\partial x^2$ on $\mathbb{T}$ is still somehow present in L_b:

Theorem Using the embedding

$$L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T}\times\mathbb{R}, \mathbb{C}^2)$$

$$f(x) \longmapsto \frac{1}{\sqrt{\pi}}\begin{bmatrix} f(x)e^{-y^2/2} \\ 0 \end{bmatrix},$$

we have

$$\lim_{b\to 0} e^{-tL_b} = e^{-t\Delta_{\mathbb{T}}/2}$$

This is a completely new phenomenon, not geometric or algebraic, but spectral.

We get the "Selberg trace formula" for π from this:

$$\sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 t} = \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} e^{-\pi^2 n^2/4t}$$

(due to Jacobi, of course, but . . .)

Thank You!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