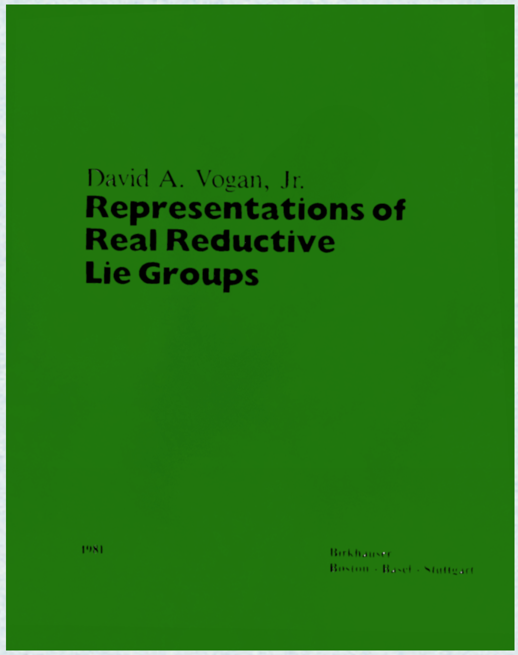
An Index Theorist Reads the Green Book

- Here is the Green Book:



- I want to explain how to use it to label the topologically non-trivial components of the tempered dual.

- Example of a topologically non-trivial component: a single discrete series representation.

- I also want to make you feel better about the (reduced) C^*-algebra of a real reductive group, and about the people who think about it.

- Speaking of whom ... I have been thinking about $C^*_r(G)$, and reading the Green Book, with a number of my NCG friends —— Pierre Clare, Tyrone Crisp, Angel Roman, Yanli Song, Xiang Tang, ...

- And thank you to David Vogan!

The Connes-Kasparov Isomorphism and Vogan's Theorem

The general aim is to obtain a statement that is beautiful, simple and useful (although beauty, and everything else, is in the eye of the beholder).

An example of such a statement:

Theorem (David Vogan) There is a bijection from $\widehat{K}$ to tempered irreducible representations of G with real infinitesimal character (given by minimal K-types).

K = max. cpt subgroup of G

This gives a "mostly" one-to-one map from $\widehat{K}$ onto the set of components of the tempered dual.

Another (clearly related, somehow):

Assume G is connected real reductive

Theorem (Lafforgue et al) There is a bijection from genuine, irreducible representations of the spin double cover of K to topologically nontrivial components of the tempered dual given by Dirac cohomology / index of the Dirac operator.

About the spin double cover:

$$\boxed{\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}}$$

$$\begin{tikzcd} \widetilde{K} \arrow[r] \arrow[d] & \mathrm{Spin}(\mathfrak{s}) \arrow[d] \\ K \arrow[r, "\mathrm{Ad}"'] & SO(\mathfrak{s}) \end{tikzcd}$$

In favorable situations $\widetilde{K} \cong K \times \mathbb{Z}_2$, and then "irreducible genuine reps of the spin double cover" correspond to "irreducible reps of K".

In this case the theorem gives a one-to-one map from $\widehat{K}$ to "most" components of the tempered dual of G.

Lafforgue's theorem is almost as close as I can get to a representation-theoretic formulation of the Connes-Kasparov isomorphism (part of the Baum-Connes conjecture). I'll improve upon it a bit by explaining what "topologically non-trivial" means (in two different ways — one of them representation-theoretic).

But first, some remarks on $C^*_r(G)$...

Everyone knows the definition of $C^*_r(G)$:

$$C^*_r(G) = \begin{array}{l}\text{norm-completion of } C_c^\infty(G), \\ \text{or } L^1(G) \text{ as bounded} \\ \text{convolution operators on } L^2(G).\end{array}$$

But the definition does not necessarily reveal much...

The way to understand $C^*_r(G)$ is via a sort of Paley-Wiener theorem.

- Each tempered admissible unitary representation of G, $\pi: G \longrightarrow U(H)$ induces

$$\pi: C^*_r(G) \longrightarrow \mathcal{K}(H)$$

COMPACT operators on H (thanks to admissibility)

$$\pi(f) = \int_G f(g)\pi(g)\,dg$$

- Such reps π arise in families described by discrete parameters σ and continuous parameters $\varphi \in \mathfrak{a}^*_\sigma$

$\boxed{\text{The parameter space for } \sigma \text{ fixed — a vector space}}$

We obtain

$$\pi_\sigma : C^*_r(G) \longrightarrow C_0(\mathfrak{a}^*_\sigma, \mathcal{K}(H))$$

Norm-continuous functions vanishing at infinity (Riemann-Lebesgue lemma)

There are some (Knapp-Stein) intertwining operators acting between representations in each continuous family

$$\pi_\sigma : C^*_r(G) \longrightarrow C_0(\mathfrak{a}^*_\sigma, \mathcal{K}(H))^{W_\sigma}$$

And that's about it:

$$\oplus \pi_\sigma : C^*_r(G) \xrightarrow{\cong} \bigoplus_\sigma C_0(\mathfrak{a}^*_\sigma, \mathcal{K}(H))^{W_\sigma}$$

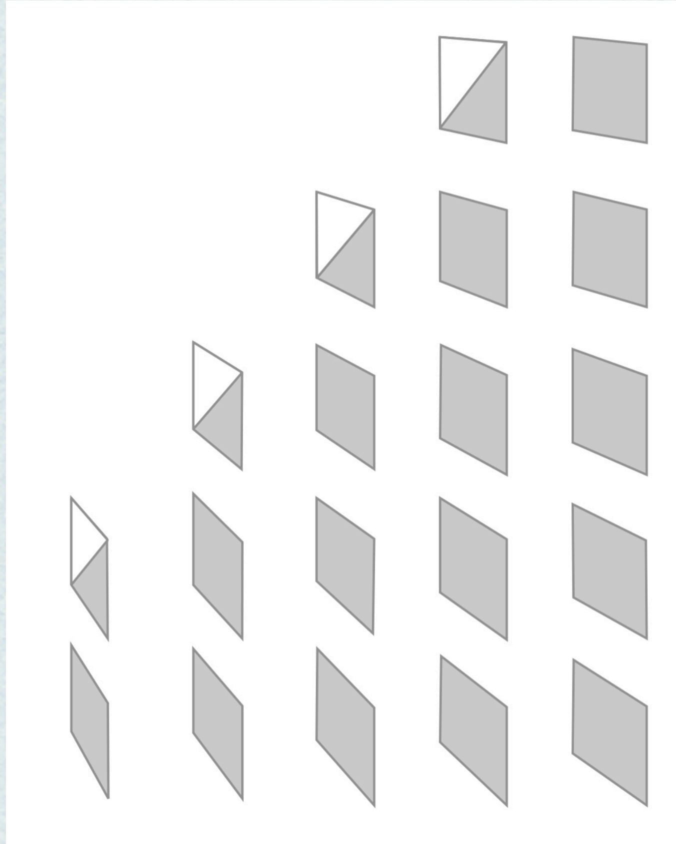
c_0-direct sum, thanks to UNIFORM ADMISSIBILITY

As a result:

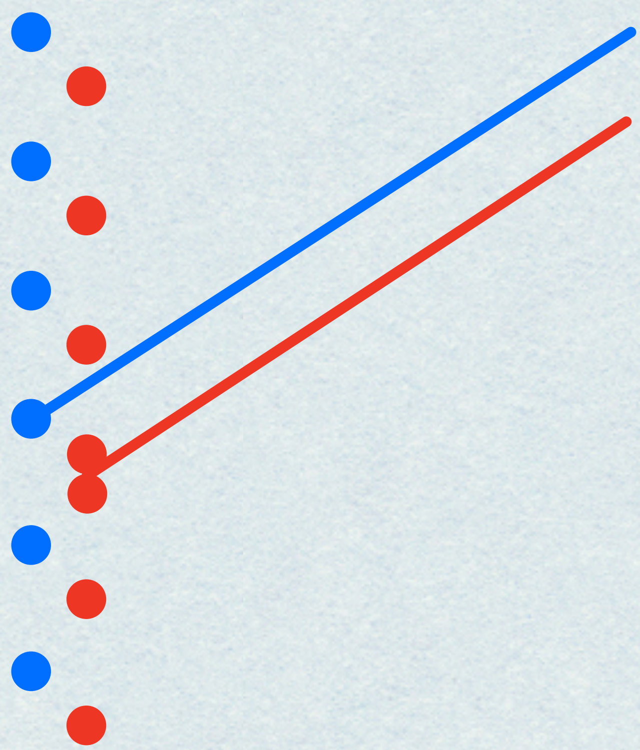
$$\boxed{C^*_r(G) \cong \text{Matrix-valued continuous } C_0\text{-functions on } \bigsqcup_\sigma \mathfrak{a}^*_\sigma / W_\sigma}$$

Here is the space (spectrum) underlying $C^*_r(G)$ for $G = GL(2,\mathbb{C})$.

(It is the tempered dual of $GL(2,\mathbb{C})$.)



And here is the space (spectrum, or tempered dual) underlying $C^*_r(G)$ for $G = SL(2,\mathbb{R})$.



A Remark on Another Convolution Algebra

- $R(\mathfrak{g}, K)$ = convolution algebra of (K-finite) distributions on G, supported on K

This is an item from algebra (despite the definition) since

≅ regular functions on $K_{\mathbb{C}}$

- $$R(\mathfrak{g}, K) \cong \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{k})}{\otimes} C^\infty(K)_{fin}$$

Its "topological" or "homological" invariants are much closer to K than G. For instance (I'm pretty sure that)

$$HP_*(R(\mathfrak{g}, K)) \cong HP_*(C^\infty(K)_{fin})$$

Periodic cyclic homology (a proxy for topological K-theory)

Using functional-analytically-defined convolution algebras brings one much closer to the representation theory of G.

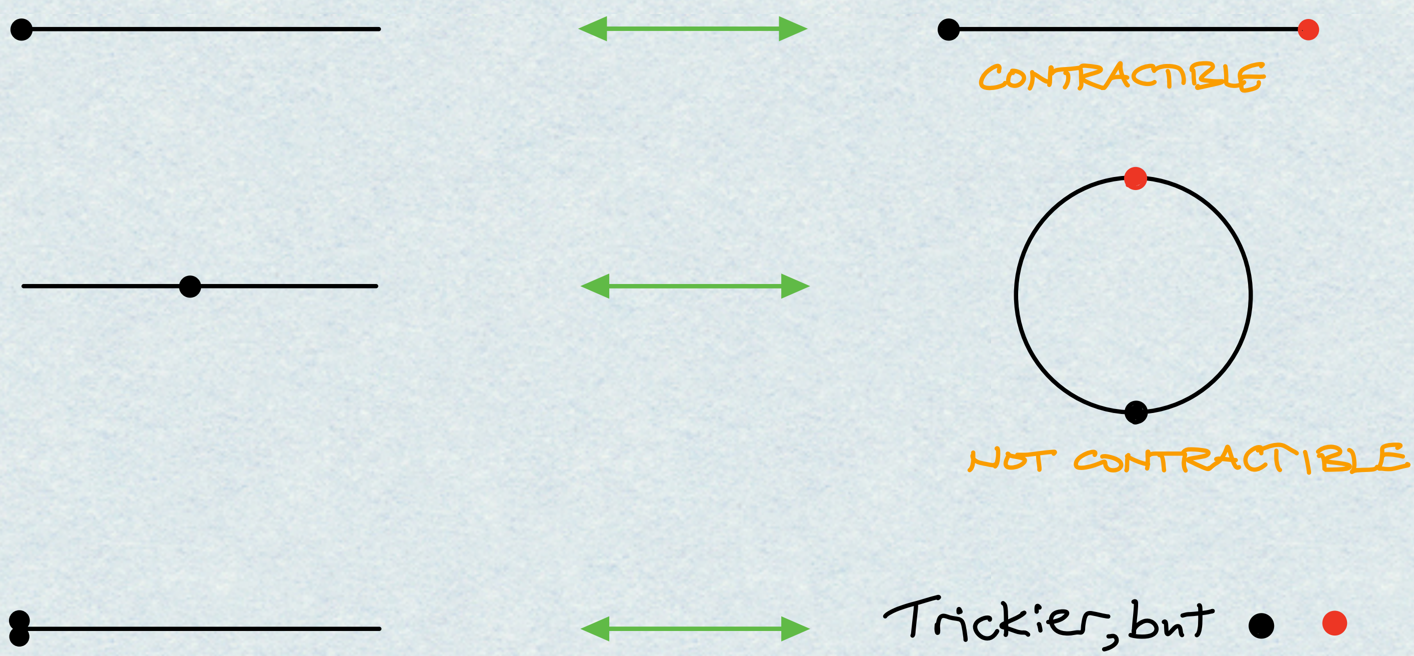
It would be interesting to examine other convolution algebras along the axis from "very algebraic" to "very functional-analytic."

Topologically Non-Trivial Components

Back to $C^*_r(G)$ and the Paley-Wiener theorem. Because of the Riemann-Lebesgue lemma, it is appropriate to view each component of the tempered dual within the homotopy category of locally compact spaces (at least, this is what K-theory does). Now

$$\begin{pmatrix}\text{Locally compact}\\ \text{spaces}\end{pmatrix} \underset{\text{equiv.}}{\simeq} \begin{pmatrix}\text{Pointed compact}\\ \text{spaces}\end{pmatrix}$$

$$\text{Locally compact spaces} \underset{\text{remove the base point}}{\overset{\text{one-point compactification}}{\rightleftarrows}} \text{Pointed compact spaces}$$



The space on the LHS is "slightly non-commutative" and is handled using NCG & K-theory.

K-Theory of Components

Knapp & Stein

Wassermann proved, using $W_\sigma = W'_\sigma \rtimes R_\sigma$ that:

<u>Theorem</u> Let σ be a parameter labelling a component of the tempered dual.

- If $W'_\sigma \neq e$, then the component is K-theoretically trivial
- If $W'_\sigma = e$, then the component is K-theoretically equivalent to a point (with a dimension shift).

So $K(C^*_r(G))$ is a free abelian group on the set of components σ with $W'_\sigma = e$.

What is that?

Time to Read the Green Book (Finally)

It seems to be an enormous challenge to determine the set of topologically non-trivial summands from the Harish-Chandra, Knapp-Stein, Knapp-Zuckerman point of view...

But using David's point of view (as detailed in the Green Book) there is a very simple description of this set —— and the parameters used are precisely those of Lafforgue's theorem (the Connes-Kasparov isomorphism).

Fix a maximal torus in K, and a system of positive roots for $(\mathfrak{k}, \mathfrak{t})$.

The irreducible genuine reps of $\tilde{K}$ correspond to dominant shifted-integral weights $\kappa \in i\mathfrak{t}_0^*$.

According to David, the components of the tempered dual (all of them) correspond to (K-conjugacy classes of) sets of Vogan data $(\mathfrak{q}, H, \delta)$.

Let L = normalizer of $\mathfrak{q}$ in G. The correspondence is

$$(\mathfrak{q}, H, \delta) \xrightarrow[\text{parabolic induction}]{} \begin{array}{c}\text{Principal series}\\ \text{rep'n of } L\end{array} \xrightarrow[\text{cohomological induction}]{} \begin{array}{c}\text{Rep'n } X(\mathfrak{q}, H, \delta)\\ \text{of } G\end{array}$$

Theorem There is a simple, explicit construction

$$\begin{pmatrix}\text{shifted-integral} \\ \text{dominant weights}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{sets of} \\ \text{Vogan data}\end{pmatrix}$$

that induces a bijection to the set of topologically non-trivial components of the tempered dual of G.

- κ = shifted integral dominant weight for $(\mathfrak{k}, \mathfrak{t})$
- $\lambda = \kappa + \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}))$ $\quad (\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s})$
- $\mu = \kappa - \rho(\Delta^+(\mathfrak{s}, \mathfrak{t}))$ $\quad$ (uses choice of $\Delta^+(\mathfrak{g}, \mathfrak{t})$)

$$\boxed{\mathfrak{q} = \bigoplus_{\langle \alpha, \lambda \rangle \geq 0} \mathfrak{g}_\alpha \qquad \delta = \exp(\mu)\big|_{H \cap T}}$$

(independent of above choice)

Remark In the equal rank case

- Typically, $\mathfrak{q}$ = Borel subalgebra

  $X(\mathfrak{q}, H, \delta)$ = discrete series

  κ = Harish-Chandra parameter

- Exceptionally, $\mathfrak{q} \cap \bar{\mathfrak{q}} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \cdots \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{z}$

Thank You!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