I plan to describe a problem in K-theory of C^*-algebras (and other algebras) that has an interesting interpretation in representation theory (I think).

Some of this is joint work with Jacob Bradd. (The rest is entirely due to him.)

- MINI-series
- MINI-lectures
- MINImal prerequisites

In brief, I want to work my around a diagram of K-theory groups (for $G = SL(2,\mathbb{R})$):

Casselman's Schwartz algebra

Connes & Kasparov

Harish-Chandra's Schwartz algebra

$$
\begin{tikzcd}
K(\mathcal{S}(G)) \arrow[r, "\cong"] \arrow[d, "\cong"', "\text{Paley-Wiener}"'] & K(\mathcal{C}(G)) \\
K(A(G)) \arrow[r, "\cong"'] & K(B(G)) \arrow[u, "\cong"]
\end{tikzcd}
$$

Thanks to Langlands (?)

Image under Fourier transform

Oka Principle

About K-Theory

The quickest definition of K-theory (of a Banach or Fréchet algebra) is also the most appropriate one for this talk.

A = Banach algebra

The Fréchet case involves some modifications

$GL_n(A)$ = ∞-dimensional topological group of invertible elements in $M_n(A)$

$$K_j(A) = \lim_{\overrightarrow{n}} \pi_{j-1}(GL_n(A))$$

Bear with me, please!

First of all, Bott's famous periodicity theorem carries over to this context:

$$K_j(A) \cong K_{j+2}(A).$$

Secondly, the new context incorporates the topological K-theory of Atiyah & Hirzebruch (made from vector bundles):

$$K^{-j}(X) \cong K_j(C(X)).$$

(A map $X \longrightarrow GL(n,\mathbb{C})$ determines a rank n vector bundle over the suspension SX by "clutching".)

One Page About Baum-Connes, Novikov

I'm mostly interested in convolution algebras on groups, like

$$C^*_r(G) \subseteq \mathcal{B}(L^2(G))$$

When G is discrete, it is very difficult to say anything useful about the unitary representation theory of G. But K-theory says something (if you can compute it). And actually it is useful, e.g. in manifold theory, where $G = \pi_1(M)$.

There is an assembly map

$$\mu : K^{-j}(T^*M) \longrightarrow K_j(C^*_r(G))$$

that is analogous to Atiyah & Singer's index homomorphism

$$\mathrm{ind} : K^0(T^*M) \longrightarrow \mathbb{Z}.$$

an isomorphism for contractible manifolds

For M = surface, genus ≤ 0, and many other aspherical manifolds, μ is an isomorphism (the Baum-Connes isomorphism for $G = \pi_1(M)$).

But what about (real reductive) Lie groups, where we know a lot about unitary representations (and even more about tempered representations)?

Is there any point in studying $K(C_r^*(G))$?

- K-theory statements tend to be very simple (intricate representation theory details don't translate into K-theory very well).
- They tend to be very general.
- They tend to be approachable from multiple directions.

I'll try to illustrate with an example that I hope will be a little bit interesting from a representation theory point of view.

However...

K-theory of representations

At present, this is a non-existent mathematical region. However, it may really come into existence in a near future.

Kasparov
1983 ICM address)

Convolution Algebras

$\mathcal{S}(G)$ = Casselman's Schwartz algebra of (very) rapidly decreasing functions on a reductive group (the natural algebra of Schwartz functions on G, viewed as a Nash manifold).

$\mathcal{C}(G)$ = Harish-Chandra's L^2-Schwartz algebra. Its K-theory is the same as that of $C^*_r(G)$.

How does the inclusion $\mathcal{S}(G) \longrightarrow \mathcal{C}(G)$ look in K-theory?

$$K_*(\mathcal{S}(G)) \longrightarrow K_*(\mathcal{C}(G))$$

Actually we know this map is an isomorphism, thanks to Connes-Kasparov-Lafforgue. But that's a long, technical, index-theoretic story... are there other (representation-theoretic) perspectives?

First of all, it is not a simple issue!

π = discrete series representation

$$p(g) = d_\pi \cdot \langle v, \pi(g^{-1})v \rangle$$

(normalized) matrix coefficient fn.

This is an idempotent in $\mathcal{C}(G)$, and (Schur orthogonality)

$$t \longmapsto e^{2\pi i t p} \qquad (t \in [0,1])$$

is a loop in $GL_1(\mathcal{C}(G))$. But p can't be moved through idempotents to an element in $\mathcal{S}(G)$. Far from it.

In fact there is nothing in $K_0(\mathcal{S}(G))$ that is obviously related to the discrete series representation π.

Paley – Wiener Theorems

I'm going to work with $G = SL(2, \mathbb{R})$, which is one of the few examples where everything is worked out (Jacob did it).

We can get a picture of $\mathcal{S}(G)$ by representing Casselman's algebra in the (non-unitary) principal series.

To simplify, fix a finite set $S \subseteq \widehat{SO(2)}$ and form

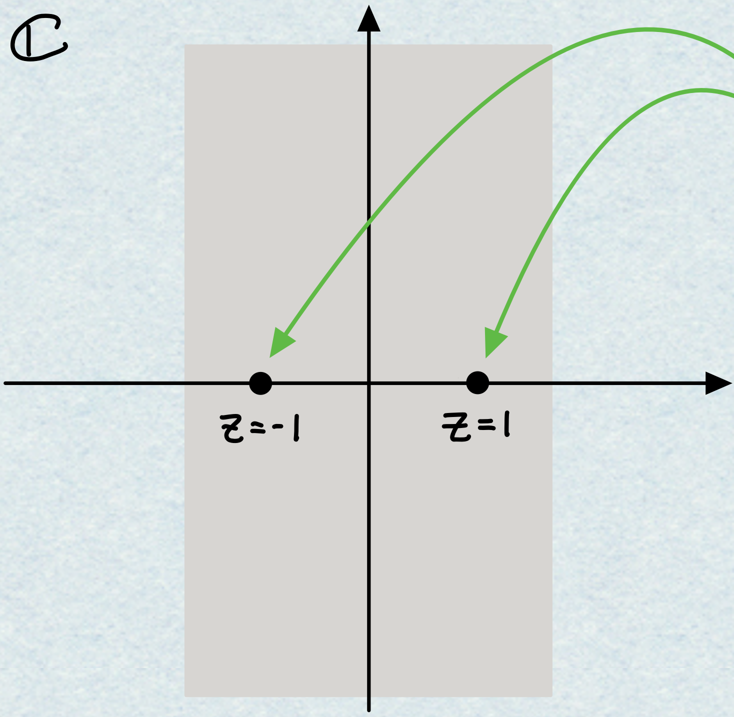
$$\mathcal{S}(G, S) = p_S \cdot \mathcal{S}(G) \cdot p_S \subseteq \mathcal{S}(G)$$

projection onto the combined isotypical subspaces for the $SO(2)$-reps in the set S.

This algebra is represented on the combined S-isotypical subspaces in any principal series representation.

Let's take $S = \{-2, 0, 2\}$. Then the action of $\mathcal{S}(G, S)$ on the spherical principal series gives ...

$$\mathcal{S}(G,S) \cong \left\{ f : \mathbb{C} \to M_3(\mathbb{C}) : \begin{array}{l} \bullet\ \text{holomorphic} \\ \bullet\ \text{Schwartz-class on strips around the unitary p. s.} \\ \bullet\ \text{additional conditions from intertwiners} \end{array} \right\}$$



The principal series are reducible at $z=\pm 1$, with $SO(2)$-types from S spread across multiple composition factors.

The "additional conditions" are that f has the form

$$f(z) = \begin{bmatrix} f_{11}(z) & (z+1)f_{12}(z) & (z^2-1)f_{13}(z) \\ (z-1)f_{21}(z) & f_{22}(z) & (z-1)f_{23}(z) \\ (z^2-1)f_{31}(z) & (z+1)f_{32}(z) & f_{33}(z) \end{bmatrix}$$

with $f_{ij}(z) = f_{ij}(-z)$ for all z.

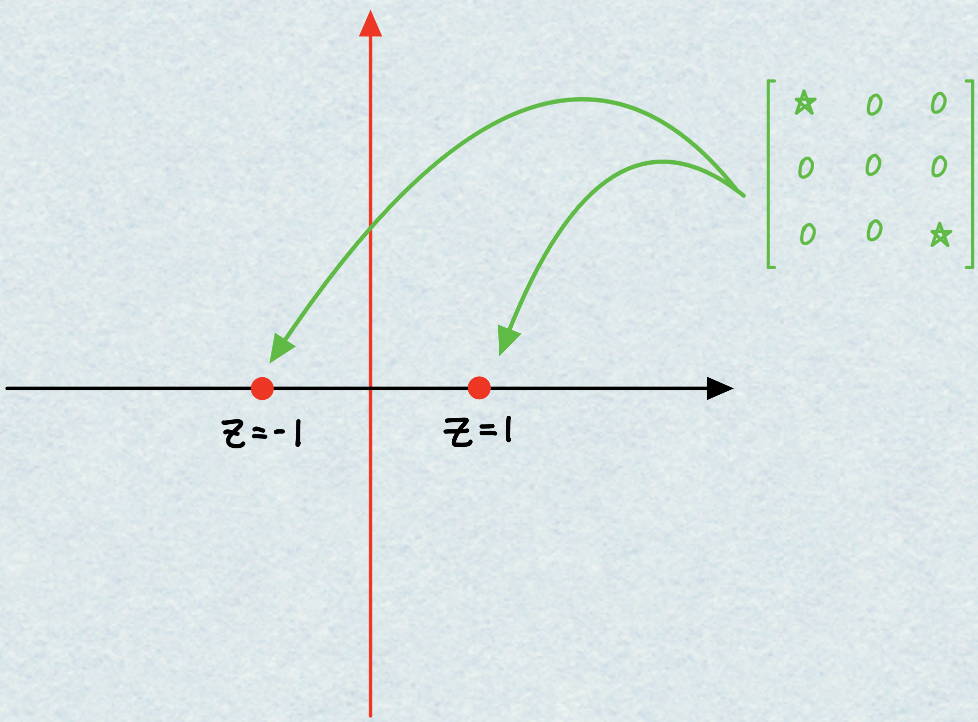
As a check, note that

$$f(1) = \begin{bmatrix} \star & \star & 0 \\ 0 & \star & 0 \\ 0 & \star & \star \end{bmatrix} \qquad f(-1) = \begin{bmatrix} \star & 0 & 0 \\ \star & \star & \star \\ 0 & 0 & \star \end{bmatrix}$$

$\mathbb{C}_{-2}$ & $\mathbb{C}_2$ are subrepresentations

$\mathbb{C}_0$ is a subrepresentation

There is a similar Paley-Wiener description of of $\mathcal{C}(G,S)$, using only the tempered parts of the non-unitary principal series.



For now, however, the two algebras $\mathcal{S}(G,S)$ and $\mathcal{C}(G,S)$ don't look very similar...

Oka Principle

This is about base change from holomorphic functions to smooth functions (following Oka, Grauert, Cartan, ...).

Theorem On a complex submanifold of $\mathbb{C}^n$, K-theory made from holomorphic vector bundles is equal to K-theory made from smooth vector bundles.

Theorem The K-theory of the algebra $A(G,S)$ above (made from holomorphic f_{ij}, Schwartz-class on strips) is equal to the K-theory of the algebra $B(G,S)$ made in exactly the same way, but from smooth f_{ij}, Schwartz-class on strips.

Not a convolution algebra!

Theorem (Novodvorskii) For any commutative Banach algebra, the Gelfand transform induces an isomorphism in K-theory.

Now, let's examine this new algebra $B(G,S)$, comprised in our example of smooth functions

$$f : \mathbb{C} \longrightarrow M_3(\mathbb{C})$$

$$f(z) = \begin{bmatrix} f_{11}(z) & (z+1)f_{12}(z) & (z^2-1)f_{13}(z) \\ (z-1)f_{21}(z) & f_{22}(z) & (z-1)f_{23}(z) \\ (z^2-1)f_{31}(z) & (z+1)f_{32}(z) & f_{33}(z) \end{bmatrix}$$

with $f_{ij}(z) = f_{ij}(-z)$. A first observation: the inclusion

$$\mathcal{S}(G,S) \longrightarrow \mathcal{C}(G,S),$$

or equivalently the inclusion

$$A(G,S) \longrightarrow \mathcal{C}(G,S),$$

extends to a surjection

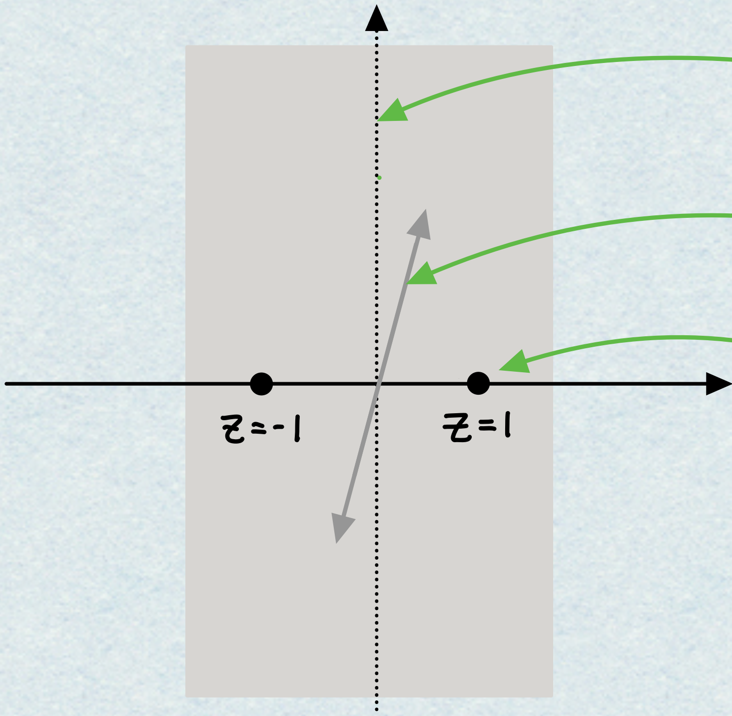
$$\boxed{B(G,S) \longrightarrow \mathcal{C}(G,S)}$$

And we ask: does this induce an isomorphism in K-theory? Here, there is a simple criterion

Isomorphism in K-theory $\Longleftrightarrow$ $K_*(\text{kernel}) = 0$

$GL_n(J) \to GL_n(B) \to GL_n(B/J)$ is a fibration

Finally, let's examine the kernel from the perspective of the Langlands classification (?)



functions in the ideal vanish on the unitary axis

$f_{ij}(z) = f_{ij}(-z)$

$$f(1) = \begin{bmatrix} 0 & \star & 0 \\ 0 & \star & 0 \\ 0 & \star & 0 \end{bmatrix}$$

<u>Theorem</u> The inclusion

$$h \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

"Langlands quotient"

is an isomorphism in K-theory from

$$\left\{ h : \mathbb{C} \to \mathbb{C} \;\middle|\; \begin{matrix}\text{Schwartz class}\\ \text{on strips}\end{matrix},\ h(z) = h(-z),\ h\big|_{i\mathbb{R}} = 0 \right\}$$

into the kernel. And the K-theory of the former is zero.

Thank You!


vogue
JACQUES
DUTRONC
MINI-MINI-MINI
$C^*_r(G)$


vogue
ROBERT LANGLANDS
MINI-MINI-MINI