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I hope these talks will be somewhat relaxing for you and me both...

There are some general rules that I shall need to follow, as best I can:

- Mini-series. A small number of talks on (possibly loosely) related topics.
- Mini-lectures. Short talks, that fit into tea times.
- Minimal prerequisites.

I'm going to talk about index theory. This uses functional analysis, as seen in the theory of linear PDE, to obtain results in topology, and in other places too.

The subject has two aspects:

- The aspect involving K-theory.
  (Atiyah, Hirzebruch, Singer, ...)
- The aspect involving the heat equation.
  (Weyl, Carleman, Pleijel, ..., McKean & Singer, ...)

I'll talk about the second aspect today. Then I'll talk about K-theory (albeit rather indirectly) in lectures 2 & 3. Then I'll return to the heat equation in the final lecture.

The Lefschetz Theorem and Index Theory

Lefschetz theory is a topic that might have led to the formulation and proof of the index theorem. (But that is not what actually happened.)

Here is the original Lefschetz theorem:

Theorem Let $f\colon M \to M$ be any smooth map on a smooth, closed manifold (or any continuous map defined on a reasonable space). If the Lefschetz number

$$\boxed{L(f) = \sum (-1)^p \operatorname{Trace}\left(f^*\colon H^p(M) \to H^p(M)\right)}$$

is nonzero, then f has a fixed point.

Moreover, if f has only nondegenerate fixed points, then $L(f)$ may be computed as a sum of contributions, one from each fixed point.

My aim is to sketch the functional analysis/heat equation proof of this result, hint at the famous extensions of the Lefschetz theorem that immediately present themselves, then look ahead to the index theorem.

The proof of Lefschetz usually (always?) proceeds as follows:

- Pick a complex that computes cohomology, and examine the action of $f: M \to M$ on it:

$$\begin{array}{ccccccc} C^0(M) & \longrightarrow & C^1(M) & \longrightarrow & C^2(M) & \longrightarrow & \cdots \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & \\ C^0(M) & \longrightarrow & C^1(M) & \longrightarrow & C^2(M) & \longrightarrow & \cdots \end{array}$$

- Use the "Euler principle"

$$\sum (-1)^p \operatorname{Trace}\left(f^*: H^p(M) \to H^p(M) \right)$$

$$= \sum (-1)^p \operatorname{Trace}\left(f^*: C^p(M) \to C^p(M) \right)$$

In the functional analysis/heat kernel proof (due to Atiyah & Bott) one uses the de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

An obvious issue is that it is infinite-dimensional but this is easy to address...

Heat Operators

Let Δ be the Laplacian on M, acting on forms (more about this later). The heat operators

$$\exp(-t\Delta) : \Omega^p(M) \longrightarrow \Omega^p(M)$$

are defined by

$$\boxed{\begin{array}{l} u_t = \exp(-t\Delta)u_0 \\ \quad (t \geq 0) \end{array} \iff \begin{cases} \dfrac{\partial u_t}{\partial t} = -\Delta u_t \\ u_0 = u \end{cases}}$$

Theorem The heat operators are represented by smooth integral kernels:

$$\big(\exp(-t\Delta)u\big)(m_1) = \int_M k_t^{(p)}(m_1, m_2)\, u(m_2)\, dm_2$$

$$(u \in \Omega^p(M))$$

This is plausible from the point of view of "physics," and proveable using the ellipticity and positivity of Δ.

More on Lefschetz and the Euler Principle

Lemma There is a chain homotopy

$$I \sim \exp(-t\Delta) : \Omega^*(M) \longrightarrow \Omega^*(M)$$

It is now legal and (mostly) correct to write

$$L(f) = \sum (-1)^p \operatorname{Trace}\left(f^* \circ \exp(-t\Delta) : H^p(M) \to H^p(M)\right)$$

$$= \sum (-1)^p \operatorname{Trace}\left(f^* \circ \exp(-t\Delta) : \Omega^p(M) \to \Omega^p(M)\right).$$

$$\therefore \quad \boxed{L(f) = \sum (-1)^p \int_M tr\left(\Lambda^p D_m^* f \circ k_t^{(p)}(f(m), m)\right) dm}$$

This uses:

- $f^* \circ \exp(-t\Delta) : \Omega^p(M) \to \Omega^p(M)$ has integral kernel $k_t^{(p)}(f(m_1), m_2)$.
  Or, to be more precise, the above, where
  $$\Lambda^p D_m^* f : \Lambda^p T^*_{f(m)} M \longrightarrow \Lambda^p T^*_m M$$

- The trace (following Hilbert) is the integral above.

As for the proof of the lemma (on the existence of a chain homotopy) we need to solve

$$d \cdot \boxed{} + \boxed{} \cdot d = I - \exp(-t\Delta)$$

At this point, I need to say that the Laplacian on forms is defined by the formula

$$\Delta = d^* d + d d^*$$

This makes the following guess reasonable, and and in fact correct:

$$\boxed{} = d^* \cdot \frac{I - \exp(-t\Delta)}{\Delta}$$

(The various "functions" of Δ can be defined by the functional calculus for operators. For instance

$$\exp(-t\Delta) = \frac{1}{2\pi i} \int e^{-t\lambda} (\lambda - \Delta)^{-1} \, d\lambda$$

Admittedly, there are high overhead costs to be paid here, but we are almost done...)

First Dividend

The problem now is to say something intelligent about

$$\boxed{\begin{aligned}&\mathrm{Trace}\left(f^*\circ\exp(-t\Delta):\Omega^p(M)\to\Omega^p(M)\right)\\ &\qquad\qquad = \int_M \mathrm{tr}\left(\Lambda^p D_m^* f\circ k_t^{(p)}(f(m),m)\right)\,dm\end{aligned}}$$

For this, back to the heat equation...

Theorem As $t\to 0$,

$$k_t(m_1,m_2)\sim t^{-n/2}\,e^{-d(m_1,m_2)^2/t}$$

(some constants and one other detail have been suppressed — $k_t(m_1,m_2)$ should be an operator $\Lambda^p T^*_{m_2}M \longrightarrow \Lambda^p T^*_{m_1}M$).

So if $f(m)\neq m$ the integrand in $\boxed{}$ above converges sharply to 0 as $t\to 0$.

First conclusion: no fixed points $\Rightarrow L(f)=0$.

But it is more revealing for index theory to compute the contributions from individual (isolated) fixed points...

Isolated Fixed Points

As $t \to 0$,

$$\int_M tr\left(\Lambda^p D_m^* f \circ k_t^{(p)}(f(m), m)\right) dm$$

$$\sim \sum_{\substack{\text{fixed} \\ \text{points } q}} \int_{\substack{\text{Neighborhood} \\ \text{of } q}} tr\left(\Lambda^p D_m^* f \circ k_t^{(p)}(f(m), m)\right) dm$$

$$\sim \sum_{\substack{\text{fixed} \\ \text{points } q}} tr\left(\Lambda^p D_q^*\right) \cdot t^{-n/2} \int_{T_q M} e^{-\|A_q X\|^2/t} \, dX$$

$\|A_q X\|^2 \approx d(f(m), m)^2$
for $m = \exp_q(X)$

suppressing constants again, where

$$A_q = I - D_q f : T_q M \to T_q M$$

The Gaussian integrals can be computed exactly, leading to

$$\int_M tr\left(\Lambda^p D_m^* f \circ k_t^{(p)}(f(m), m)\right) dm \sim \sum_{\substack{\text{fixed} \\ \text{points } q}} \frac{\operatorname{Trace}\left(\Lambda^p D_q^* f\right)}{\left|\det(I - D_q f)\right|}$$

Adding up over all p, using

$$\det\left(I - D_q f\right) = \sum_{p=0}^{n} (-1)^p \operatorname{Trace}\left(\Lambda^p D_q^* f\right)$$

gives ...

Theorem In the case of isolated non-degenerate fixed points,

$$L(f) = \sum_{\substack{\text{fixed} \\ \text{points } q}} \frac{\det(I - D_q f)}{|\det(I - D_q f)|}$$

This is the usual Lefschetz result. What made the argument that I sketched newsworthy is that it extends readily to the complex case — to the Dolbeault cohomology of a holomorphic bundle E, where it gives

$$L^{\text{Dolbeault}}(f) = \sum_{\substack{\text{fixed} \\ \text{points } q}} \frac{\operatorname{Trace}(f_q : E_q \to E_q)}{\det_{\mathbb{C}}(I - D_q f)}$$

... the formula seemed too beautiful to be wrong ...

We were especially convinced when one day we suddenly realized that the famous Hermann Weyl character formula was a particular case of our general formula.

Michael Atiyah

Notes for Index Theory

- The argument relies on the fact that each term

$$\mathrm{Trace}\left(f^*\circ \exp(-t\Delta) : \Omega^p(M) \to \Omega^p(M)\right)$$

($p = 0, 1, 2, \dots$) converges to a finite limit as $t \to 0$. If the fixed point set F is higher-dimensional, then this is no longer true:

$$\mathrm{Trace}\left(f^*\circ \exp(-t\Delta) : \Omega^p(M) \to \Omega^p(M)\right) \sim t^{-\dim(F)/2}$$

- These divergences cancel out (since the alternating sum defining $L(f)$ is independent of t). But the "fantastic cancellations" that are responsible for this are hard to fathom.

- Of course, the index problem, where $f = id$ and $F = M$ is hardest of all...

- For the experts, it is interesting to remember that f need not preserve any Riemannian structure (making index theory diffeomorphism invariant continues to be a matter of interest).