Archimedean period relations and period relations for automorphic L-functions

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Riemann zeta function:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots .$$

Properties:

- Absolutely converges when $\mathrm{Re}(s) > 1$.
- Meromorphic continuation to $\mathbb{C}$.
- Euler factorization:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

- Gamma function:

$$\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) = \pi^{-\frac{s}{2}} \int_0^\infty e^{-t} t^{-\frac{s}{2}} \frac{dt}{t}.$$

Analogue of :

$$\Gamma_p(s) := \frac{1}{1 - p^{-s}}.$$

- Completed Riemann zeta function:

$$\Lambda(s) := \Gamma_{\mathbb{R}}(s) \cdot \zeta(s)$$

- Functional equation:

$$\Lambda(s) = \Lambda(1 - s).$$

Euler:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6},$$

$$\zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90},$$

and in general,

$$\frac{\zeta(2)}{\pi^2}, \frac{\zeta(4)}{\pi^4}, \frac{\zeta(6)}{\pi^6}, \cdots \in \mathbb{Q}.$$

Functional equation:

$$\zeta(-1), \zeta(-3), \zeta(-5), \cdots \in \mathbb{Q}.$$

Definition: Critical places of $\zeta(s)$

Integers at which neither $\Gamma_{\mathbb{R}}(s)$ nor $\Gamma_{\mathbb{R}}(1-s)$ has a pole.

They are

$$2, 4, 6, 8, \cdots$$

and

$$-1, -3, -5, -7, \cdots .$$

Question:

Higher degree L-functions?

Rationality of other L-functions

- $\mathrm{GL}(2)$: Mannin,Shimura, Harder, Hida, ...
- Motivic L-function: Conjectured by Deligne
- Rankin-Selberg L-function: Conjectured by Blasius
- Work of many authors: Schmidt, Kazhdan-Mazur-Schmidt, Kasten-Schmidt, Mahnkopf, Raghuram, Raghuram-Shahidi, Grobner-Harris, Harris-Lin, Grobner-Raghuram, Harder-Raghuram, Januszewski, Grobner-Lin $\cdots$

Main tool: Modular symbols

2. Modular symbols

- $G/_{\mathbb{Q}}$: connected reductive algebraic group
- $H \subset G$: connected algebraic subgroup
- $\chi : [H] := H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathbb{C}^\times$: automorphic character
- $\Pi \subset \mathcal{A}([G])$: irreducible automorphic representation
- $A := A_G(\mathbb{R})^\circ \cap H(\mathbb{R})$, where A_G is the maximal central split torus
- Suppose A acts on Π via the character $\chi^{-1}|_A$

Period integral:

$$Z : \Pi \otimes \chi \otimes \mathcal{M}(H) \to \mathbb{C}, \quad \phi \otimes 1 \otimes dx \mapsto \int_{A\backslash[H]} \phi(x)\chi(x)\, dx,$$

where

$$\mathcal{M}(H) := \{\text{right invariant measure on } A\backslash H(\mathbb{A})\}$$

Factorization (in some cases)

$$Z = \text{L-function} \cdot \bigotimes_v Z_v^\circ,$$

where

$$0 \neq Z_v^\circ \in \mathrm{Hom}_{\mathsf{H}(\mathbb{Q}_v)}(\Pi_v \otimes \chi_v \otimes \mathcal{M}_v, \mathbb{C}).$$

Example: Rankin-Selberg L-function

- $$\mathsf{H} = \mathrm{GL}(n-1) \subset \mathsf{G} = \mathrm{GL}(n) \times \mathrm{GL}(n-1), \qquad g \mapsto \left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}, g \right)$$

Example: Standard L-function

- $$\mathsf{H} = \mathrm{GL}(n) \times \mathrm{GL}(n) \subset \mathsf{G} = \mathrm{GL}(2n), \qquad (g_1, g_2) \mapsto \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}.$$

- $\mathsf{G}(\mathbb{C}) \curvearrowright F$: irreducible algebraic representation
- $\mathcal{X}_{\mathsf{G}} := A\backslash[G]/K_\infty^\circ$, where $K_\infty \subset \mathsf{G}(\mathbb{R})$ is a maximal compact subgroup
- Betti cohomology

$$H^*_\Phi(\mathcal{X}_{\mathsf{G}}, F^\vee) := \varinjlim_{K_f} H^*_\Phi(\mathcal{X}_{\mathsf{G}}/K_f, F^\vee)$$

DeRham cohomology$\rightarrow$ Betti cohomology

$$\mathrm{H}^*_{\mathrm{ct}}(A\backslash \mathsf{G}(\mathbb{R})^\circ, \Pi \otimes F^\vee) \rightarrow H^*_\Phi(\mathcal{X}_{\mathsf{G}}, F^\vee).$$

- $\mathsf{H}(\mathbb{C}) \curvearrowright \chi_{\mathrm{alg}}$: algebraic character
- $\eta_F \in \mathrm{Hom}_{\mathsf{H}(\mathbb{C})}(F^\vee \otimes \chi_{\mathrm{alg}}^{-1}, \mathbb{C})$
- Suppose $C_\infty := K_\infty \cap \mathsf{H}(\mathbb{R})$ is a maximal compact subgroup.
- $\mathcal{O}_{\mathsf{H}} := \mathbb{C} \cdot \text{invariant orientation on } \mathsf{H}(\mathbb{R})/AC_\infty^\circ$.

Modular symbol

$$\begin{aligned}
\mathcal{P}: \quad & \mathrm{H}^*_{\mathrm{ct}}(A\backslash \mathsf{G}(\mathbb{R})^\circ, \Pi \otimes F^\vee) \otimes \mathrm{H}^*_{\mathrm{ct}}(A\backslash \mathsf{H}(\mathbb{R})^\circ, \chi \otimes \chi_{\mathrm{alg}}^{-1}) \otimes \mathcal{M}^\sharp(\mathsf{H}) \\
\rightarrow \quad & H^*_{\Phi}(\mathcal{X}_{\mathsf{G}}, F^\vee) \otimes H^*(\mathcal{X}_{\mathsf{H}}, \chi_{\mathrm{alg}}^{-1}) \otimes \mathcal{M}^\sharp(\mathsf{H}) \\
\xrightarrow{\eta_F} \quad & \mathrm{H}^*_{\mathrm{c}}(\mathcal{X}_{\mathsf{H}}, \mathbb{C}) \otimes \mathcal{M}^\sharp(\mathsf{H}) \\
\xrightarrow{\int} \quad & \mathbb{C},
\end{aligned}$$

where

$$\mathcal{M}^\sharp(\mathsf{H}) := \mathcal{O}_{\mathsf{H}} \otimes \mathcal{M}(\mathsf{H}(\mathbb{A}_f)).$$

$\mathcal{P}$ is rational $\Rightarrow$ arithmetic of L-function

- $\mathsf{G}(\mathbb{R}) \curvearrowright \Pi_\infty$: irreducible Casselman-Wallach representation
- $\mathsf{H}(\mathbb{R}) \curvearrowright \chi_\infty$: character
- $0 \neq Z_\infty^\circ \in \mathrm{Hom}_{\mathsf{H}(\mathbb{R})}(\Pi_\infty \otimes \chi_\infty \otimes \mathcal{M}(A\backslash \mathsf{H}(\mathbb{R})), \mathbb{C})$.

Modular symbol at infinity

$$\begin{aligned} \mathcal{P}_\infty : \quad & \mathrm{H}^*_{\mathrm{ct}}(A\backslash \mathsf{G}(\mathbb{R})^\circ, \Pi_\infty \otimes F^\vee) \otimes \mathrm{H}^*_{\mathrm{ct}}(A\backslash \mathsf{H}(\mathbb{R})^\circ, \chi_\infty \otimes \chi_{\mathrm{alg}}^{-1}) \otimes \mathcal{O}_{\mathsf{H}} \\ \to \quad & \mathrm{H}^*_{\mathrm{ct}}(A\backslash \mathsf{H}(\mathbb{R})^\circ, \mathcal{M}(A\backslash \mathsf{H}(\mathbb{R}))^*) \otimes \mathcal{O}_{\mathsf{H}} \\ \to \quad & \mathbb{C} \end{aligned}$$

Factorization (in some cases)

$$\mathcal{P} = \text{special values of L-function} \cdot \mathcal{P}_\infty \otimes \bigotimes_p \mathcal{P}_p,$$

where

$$0 \neq \mathcal{P}_p \in \mathrm{Hom}_{\mathsf{H}(\mathbb{Q}_p)}(\Pi_p \otimes \chi_p \otimes \mathcal{M}(\mathsf{H}(\mathbb{Q}_p)), \mathbb{C}).$$

Goal

$$\text{understand } \mathcal{P}_\infty \text{ in the cases of interest.}$$

$$\mathcal{P}_\infty \neq 0?$$

3. Rankin-Selberg L-functions

- $\mathbb{K}$: local field
- $\psi_{\mathbb{K}} : \mathbb{K} \to \mathbb{C}^\times$: non-trivial unitary character.

Local zeta function:

$$\Gamma_{\mathbb{K}}(s) := \begin{cases} \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), & \mathbb{K} = \mathbb{R}; \\ 2(2\pi)^{-s}\Gamma(s), & \mathbb{K} \cong \mathbb{C}; \\ \frac{1}{1-q^{-s}}, & \mathbb{K} \text{ is non-archimedean.} \end{cases}$$

Irreducible "representation":

$$\mathrm{GL}_n(\mathbb{K}) \curvearrowright \Pi_{\mathbb{K}}, \quad \mathrm{GL}_m(\mathbb{K}) \curvearrowright \Sigma_{\mathbb{K}}.$$

Assume they are generic:

$$0 \neq \lambda_\Pi \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(\Pi_{\mathbb{K}}, \psi_{\mathbb{K},n}), \quad 0 \neq \lambda_\Sigma \in \mathrm{Hom}_{\mathrm{N}_m(\mathbb{K})}(\Sigma_{\mathbb{K}}, \overline{\psi_{\mathbb{K},m}}).$$

Here

$$\begin{aligned} \psi_{\mathbb{K},n} : \mathrm{N}_n(\mathbb{K}) &\rightarrow \mathbb{C}^\times, \\ [x_{i,j}] &\mapsto \psi_{\mathbb{K}}(x_{1,2} + x_{2,3} + \cdots x_{n-1,n}) \end{aligned}$$

Rankin-Selberg zeta integral ($n > m$):

$$Z(u, v, \mathrm{d}g; s) := \int_{\mathrm{N}_m(\mathbb{K})\backslash \mathrm{GL}_m(\mathbb{K})} \langle g.u, \lambda_\Pi \rangle \cdot \langle g.v, \lambda_\Sigma \rangle \cdot |\det(g)|_{\mathbb{K}}^{s-\frac{n-m}{2}} \, \mathrm{d}g,$$

where $u \in \Pi_{\mathbb{K}}$, $v \in \Sigma_{\mathbb{K}}$.

- Absolutely converges for $\mathrm{Re}(s) \gg 0$.
- Meromorphic continuation to $\mathbb{C}$.

Definition and theorem (P.S.-Jacquet-Shalika)

There exists a unique meromorphic function $\mathrm{L}(s, \Pi_{\mathbb{K}} \times \Sigma_{\mathbb{K}})$ such that

- it is a finite product of translations of $\Gamma_{\mathbb{K}}(s)$;
- the normalized zeta integral map

$$\begin{array}{rcl} \mathrm{Z}^{\circ} : \Pi_{\mathbb{K}} \times \Sigma_{\mathbb{K}} \times \mathcal{M}_{m,\mathbb{K}} \times \mathbb{C} & \to & \mathbb{C}, \\ (u, v, \mathrm{d}g; s) & \mapsto & \frac{\mathrm{Z}(u,v,\mathrm{d}g;s)}{\mathrm{L}(s,\Pi_{\mathbb{K}}\times\Sigma_{\mathbb{K}})} \end{array}$$

  is entire and nonzero on the last variable.

- $\mathcal{M}_{m,\mathbb{K}} := \{\text{invariant measure on } \mathrm{GL}_m(\mathbb{K})\}$
- Generalize to all irreducible representations
- Generalize to the case when $n = m$.

- k : number field;
- $\mathbb{A}_{\mathrm{k}} = \mathbb{A} \otimes_{\mathbb{Q}} \mathrm{k}$.

Irreducible cuspidal automorphic representations:

$$\mathrm{GL}_n(\mathbb{A}_{\mathrm{k}}) \curvearrowright \Pi = \widehat{\bigotimes_v}' \Pi_v \subset \mathcal{A}^{\infty}_{\mathrm{cusp}}(\mathrm{GL}_n(\mathrm{k})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathrm{k}})).$$

$$\mathrm{GL}_m(\mathbb{A}_{\mathrm{k}}) \curvearrowright \Sigma = \widehat{\bigotimes_v}' \Sigma_v \subset \mathcal{A}^{\infty}_{\mathrm{cusp}}(\mathrm{GL}_m(\mathrm{k})\backslash \mathrm{GL}_m(\mathbb{A}_{\mathrm{k}})).$$

Automorphic L-functions:

$$\mathrm{L}(s, \Pi \times \Sigma) := \prod_v \mathrm{L}(s, \Pi_v \times \Sigma_v).$$

- Absolutely convergence and meromorphic continuation
- Functional equation

4. Algebraic representations

- $\mathbb{K}$: archimedean local field, with algebraic closure $\overline{\mathbb{K}}$.
- $\iota, \bar{\iota} : \overline{\mathbb{K}} \to \mathbb{C}$ are the isomorphisms.

The Weil group:

$$\mathrm{W}_{\mathbb{K}} := \begin{cases} \overline{\mathbb{K}}^{\times} \sqcup \mathrm{j}\overline{\mathbb{K}}^{\times}, & \mathbb{K} = \mathbb{R}; \\ \mathbb{K}^{\times}, & \mathbb{K} \cong \mathbb{C}; \end{cases}$$

Local Langlands correspondence

$$\begin{aligned} & \mathrm{Irr}(\mathrm{GL}_n(\mathbb{K})) \\ \xleftrightarrow{r_{\mathbb{K}}} \; & \{\text{completely reducible } n\text{-dim. rep. of } \mathrm{W}_{\mathbb{K}}\}/\sim . \end{aligned}$$

Determined by:

- when $n = 1$ it is the inflation through

$$\mathrm{W}_{\mathbb{K}} \twoheadrightarrow \mathbb{K}^{\times};$$

- when $n = 2$ and $F = \mathbb{R}$,

$$\text{rel. disc. ser. of inf. char. } (a, b) \longleftrightarrow \mathrm{Ind}_{\mathbb{K}^{\times}}^{\mathrm{W}_{\mathbb{K}}} \iota^{a} \bar{\iota}^{b};$$

- parabolic induction (taking the Langlands subquotient) is compatible with direct sum.

Completely reducible n-dim. rep.:

$$\mathrm{W}_{\mathbb{K}} \curvearrowright \sigma.$$

Write

$$\sigma|_{\overline{\mathbb{K}}^\times} = \iota^{a_1}\bar{\iota}^{b_1} \oplus \iota^{a_2}\bar{\iota}^{b_2} \oplus \cdots \oplus \iota^{a_n}\bar{\iota}^{b_n}.$$

Definition (Clozel)

The representation σ is

- algebraic if all a_i's and all b_i's are integers;
- regular if a_i's are pairwise distinct, and so are b_i's;

Definition (Clozel)

An irreducible Casselman-Wallach representation $\Pi_{\mathbb{K}}$ of $\mathrm{GL}_n(\mathbb{K})$ is said to be algebraic or regula if so is

$$r_{\mathbb{K}}(\Pi_{\mathbb{K}} \otimes |\det|_{\mathbb{K}}^{\frac{1-n}{2}}).$$

- $\mathrm{GL}_n(\mathbb{A}_{\mathrm{k}}) \curvearrowright \Pi$: irreducible automorphic

Definition (Clozel)

The representation Π is algebraic or regular if so is Π_v for every $v \mid \infty$.

Suppose $\mathrm{GL}_n(\mathbb{A}_\mathrm{k}) \curvearrowright \Pi$ and $\mathrm{GL}_m(\mathbb{A}_\mathrm{k}) \curvearrowright \Sigma$ are algebraic.

Definition: Critical places of $\mathrm{L}(s, \Pi)$

A number in $\mathbb{Z} + \frac{n+m}{2}$ at which neither $\mathrm{L}(s, \Pi_v \times \Sigma_v)$ nor $\mathrm{L}(1-s, \Pi_v^\vee \times \Sigma_v^\vee)$ has a pole, for every $v \mid \infty$.

Problem (Blasius):

Arithmetic property of $\mathrm{L}(s_0, \Pi \times \Sigma)$ where s_0 is a critical palace.

5. Archimedean modular symbols

- $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n) \in \mathbb{Z}^n$: pure weight

$$\mu_1 + \mu_n = \mu_2 + \mu_{n-1} = \cdots = \mu_n + \mu_1.$$

- $\mathrm{GL}_n(\mathbb{C}) \curvearrowright F_\mu$: irreducible algebraic representation of highest weight μ
- $\mathrm{GL}_n(\mathbb{R}) \curvearrowright \pi_\mu$: irreducible Casselman-Wallach representation that is generic, essential unitarizable, and cohomological

$$\mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{R})^\circ, \pi_\mu \otimes F_\mu^\vee) \neq 0.$$

- Infinitesimal character of F_μ:

$$\tilde\mu = (\tilde\mu_1 \geq \tilde\mu_2 \geq \cdots \geq \tilde\mu_n) = \mu + (\frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{1-n}{2})$$

- If n is even, then

$$\pi_\mu \cong \mathrm{Ind}_{P_{2,2,\cdots,2}}^{\mathrm{GL}_n(\mathbb{R})} D_{\tilde\mu_1,\tilde\mu_n} \widehat\otimes D_{\tilde\mu_2,\tilde\mu_{n-1}} \widehat\otimes \cdots \widehat\otimes D_{\tilde\mu_{\frac{n}{2}},\tilde\mu_{\frac{n+2}{2}}}.$$

- If n is odd, then

$$\pi_\mu \cong \mathrm{Ind}_{P_{2,2,\cdots,2,1}}^{\mathrm{GL}_n(\mathbb{R})} D_{\tilde\mu_1,\tilde\mu_n} \widehat\otimes D_{\tilde\mu_2,\tilde\mu_{n-1}} \widehat\otimes \cdots \widehat\otimes D_{\tilde\mu_{\frac{n-1}{2}},\tilde\mu_{\frac{n+3}{2}}} \otimes (\,\cdot\,)^{\tilde\mu_{\frac{n+1}{2}}} \mathrm{sgn}^\epsilon.$$

- Clozel: These are the real components of regular algebraic cuspidal automorphic representations.

- $\psi_{\mathbb{R}} : \mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$: unitary character
- $0 \neq \lambda_\mu \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{R})}(\pi_\mu, \psi_{\mathbb{R},n})$
- $0 \neq v_\mu \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{C})}(F_\mu^\vee, \mathbb{C})$

- $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n-1}) \in \mathbb{Z}^{n-1}$: pure weight
- $\mathrm{GL}_{n-1}(\mathbb{C}) \curvearrowright F_\nu$
- $\mathrm{GL}_{n-1}(\mathbb{R}) \curvearrowright \pi_\nu$
- $0 \neq \lambda_\nu \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathbb{R})}(\pi_\nu, \overline{\psi_{\mathbb{R},n-1}})$
- $0 \neq v_\nu \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathbb{C})}(F_\nu^\vee, \mathbb{C})$

Suppose: (μ, ν) is balanced, namely for some $j \in \mathbb{Z}$

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(F_\mu^\vee \otimes F_\nu^\vee, \det^j) \neq 0.$$

Then

$$\{\text{critical place}\} = \{\frac{1}{2} + j\}$$

Rankin-Selberg integral:

$$Z_j := \frac{\mathsf{Z}(\,\cdot, \frac{1}{2}+j)}{\mathsf{L}(\frac{1}{2}+j, \pi_\mu \times \pi_\nu)} : \pi_\mu \widehat{\otimes} \pi_\nu \otimes \chi_\infty \otimes \mathcal{M}_\infty \to |\mathrm{det}|^{-j}$$

- $\chi_\infty : \mathbb{R}^\times \to \mathbb{C}^*$: finite order character
- $\mathcal{M}_\infty := \{\text{invariant measure on } \mathrm{GL}_{n-1}(\mathbb{R})\}$

- $\eta_j \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(F_\mu^\vee \otimes F_\nu^\vee, \mathsf{det}^j)$

Modular symbol at infinity

$$\begin{aligned} \mathcal{P}_{\mu,\nu,\chi_\infty,j} : \quad & \mathrm{H}^*_\mu \otimes \mathrm{H}^*_\nu \otimes \mathrm{H}^*_{\chi_\infty \otimes \mathrm{sgn}^j} \otimes \mathcal{O}_{\mathrm{GL}_{n-1}} \\ \xrightarrow{Z_j \otimes \eta_j} \quad & \mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_{n-1}(\mathbb{R})^\circ, \mathcal{M}^*_\infty) \otimes \mathcal{O}_{\mathrm{GL}_{n-1}} \\ \to \quad & \mathbb{C}. \end{aligned}$$

Here

$$\begin{aligned} \mathrm{H}^*_\mu \quad & := \quad \mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{R})^\circ, \pi_\mu \otimes F_\mu^\vee) \\ \mathrm{H}^*_{\chi_\infty \otimes \mathrm{sgn}^j} \quad & := \quad \mathsf{H}^*_{\mathrm{ct}}(\mathrm{GL}_{n-1}(\mathbb{R})^\circ, \chi_\infty \otimes \mathrm{sgn}^j) \\ & = \quad \mathsf{H}^*_{\mathrm{ct}}(\mathrm{GL}_{n-1}(\mathbb{R})^\circ, (\chi_\infty \otimes |\mathsf{det}|^j) \otimes \mathsf{det}^{-j}) \end{aligned}$$

Nonvanishing hypothesis (Kazhdan-Mazur, Sun)

$$\mathcal{P}_{\mu,\nu,\chi_\infty,j} \neq 0.$$

Problem

How does $\mathcal{P}_{\mu,\nu,\chi_\infty,j}$ vary when j varies

Similar problem

How does $\zeta(2k)$ vary when k varies

6. Archimedean period relations

- Specify $\mu \to 0_n = (0, 0, \cdots, 0) \in \mathbb{Z}^n$, we have

$$\mathrm{GL}_n(\mathbb{R}) \curvearrowright \pi_{0_n}, \quad 0 \neq \lambda_{0_n} \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{R})}(\pi_{0_n}, \psi_{\mathbb{R},n})$$

H^*_μ is independent of μ

Assume agreement of the central character. There is a unique element $\jmath_\mu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{R})}(\pi_{0_n}, \pi_\mu \otimes F_\mu^\vee)$ such that the diagram

$$\begin{array}{ccc} \pi_{0_n} & \xrightarrow{\jmath_\mu} & \pi_\mu \otimes F_\mu^\vee \\ \lambda_{0_n} \downarrow & & \downarrow \lambda_\mu \otimes v_\mu \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

commutes. Moreover, $\jmath_\mu$ induces a linear isomorphism

$$\jmath_\mu : \mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{R})^0; \pi_{0_n}) \xrightarrow{\sim} \mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{R})^0; \pi_\mu \otimes F_\mu^\vee).$$

Normalization of η_j

There is a unique element

$$\eta_j \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(F_\mu^\vee \otimes F_\nu^\vee, \det^j)$$

such that

$$\eta_j((z_n^{-1} \cdot v_\mu^\vee) \otimes (z_{n-1}^{-1} \cdot v_\nu^\vee)) = 1.$$

- $(z_n^{-1} \cdot v_\mu^\vee) \otimes (z_{n-1}^{-1} \cdot v_\nu^\vee)$ is a highest weight vector with respect to a Borel subgroup that is transversal to $\mathrm{GL}_{n-1}(\mathbb{C}) \subset \mathrm{GL}_{n-1}(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

- $v_\mu^\vee \in (F_\mu^\vee)^{\bar{\mathrm{N}}_n(\mathbb{C})}$, $\langle v_\mu, v_\mu^\vee \rangle = 1$.
- $z_k \in \mathrm{GL}_n(\mathbb{Z})$ defined by $z_1 = [1]$ and for $k \geq 2$,

$$z_k := \begin{bmatrix} w_{k-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_{k-2}^{-1} & 0 \\ 0 & 1_2 \end{bmatrix} \begin{bmatrix} {}^t z_{k-1} w_{k-1} z_{k-1} & {}^t e_{k-1} \\ 0 & 1 \end{bmatrix}$$

where

$$w_k := \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ & \cdots & \cdots & \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathrm{GL}_k(\mathbb{Z})$$

$$e_{k-1} := [0, \cdots, 0, 1] \in \mathbb{Z}^{1 \times (k-1)}$$

Archimedean period relation, Li-Liu-Sun [Preprint 2021]

$$\begin{array}{ccc} \mathrm{H}^*_\mu \otimes \mathrm{H}^*_\nu \otimes \mathrm{H}^*_{\chi_\infty \otimes \mathrm{sgn}^j} \otimes \mathcal{O}_{\mathrm{GL}_{n-1}} & \xrightarrow{\Omega_{\mu,\nu,j}\cdot \mathcal{P}_{\mu,\nu,\chi_\infty,j}} & \mathbb{C} \\ \uparrow {\scriptstyle \jmath_\mu \otimes \jmath_\nu \otimes \mathrm{id} \otimes \mathrm{id}} & & \uparrow \\ \mathrm{H}^*_{0_n} \otimes \mathrm{H}^*_{0_{n-1}} \otimes \mathrm{H}^*_{\chi_\infty \otimes \mathrm{sgn}^j} \otimes \mathcal{O}_{\mathrm{GL}_{n-1}} & \xrightarrow{\mathcal{P}_{0_n, 0_{n-1}, \chi_\infty \otimes \mathrm{sgn}^j, 0}} & \mathbb{C} \end{array}$$

$$\Omega_{\mu,\nu,j} := (\epsilon_{\psi_\mathbb{R}} \cdot \mathrm{i})^{j \cdot \frac{n(n-1)}{2}} \cdot c'_\mu \cdot c_\nu \cdot \varepsilon_{\mu,\nu},$$

$$c'_\mu := \prod_{i=1}^{n-1} ((-1)^n \cdot \epsilon_{\psi_\mathbb{R}} \cdot \mathrm{i})^{(n-i)\mu_i}, \quad c_\nu := \prod_{i=1}^{n-1} ((-1)^n \cdot \epsilon_{\psi_\mathbb{R}} \cdot \mathrm{i})^{(n-i)\nu_i},$$

$$\varepsilon_{\mu,\nu} := \prod_{k<i,\, k+i \le n} (-1)^{\mu_i + \nu_k}, \quad \epsilon_{\psi_\mathbb{R}} = \pm 1.$$

Similar for $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n-1}(\mathbb{C})$

7. Period relations for Rankin-Selberg L-functions

- $\mathrm{GL}_n(\mathbb{A}_\mathrm{k}) \curvearrowright \Pi$, $\mathrm{GL}_{n-1}(\mathbb{A}_\mathrm{k}) \curvearrowright \Sigma$: irreducible cuspidal automorphic, regular algebraic. Balanced coefficient systems.
- : $\chi : \mathrm{k}^\times \backslash \mathbb{A}_\mathrm{k}^\times \to \mathbb{C}^\times$: finite order character
- $\frac{1}{2} + j$: critical place

Theorem (Li-Liu-Sun, preprint, 2021)

$$\frac{\mathrm{L}(\frac{1}{2}+j, \Pi \times \Sigma \times \chi)}{\Omega_{\mu,\nu,j} \cdot \mathcal{G}(\chi_\Sigma) \cdot \mathcal{G}(\chi)^{\frac{n(n-1)}{2}} \cdot \Omega_{\chi_\infty^{(j)}\epsilon_\infty}(\Pi) \cdot \Omega_{\chi_\infty^{(j)}\epsilon_\infty}(\Sigma)} \in \mathbb{Q}(\Pi, \Sigma, \chi)$$

- For every $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\sigma\left(\frac{\mathrm{L}(\frac{1}{2}+j, \Pi \times \Sigma \times \chi)}{\Omega_{\mu,\nu,j} \cdot \mathcal{G}(\chi_\Sigma) \cdot \mathcal{G}(\chi)^{\frac{n(n-1)}{2}} \cdot \Omega_{\chi_\infty^{(j)}\epsilon_\infty}(\Pi) \cdot \Omega_{\chi_\infty^{(j)}\epsilon_\infty}(\Sigma)}\right)$$
$$= \frac{\mathrm{L}(\frac{1}{2}+j, {}^\sigma\Pi \times {}^\sigma\Sigma \times {}^\sigma\chi)}{\Omega_{\mu,\nu,j} \cdot \mathcal{G}(\chi_{{}^\sigma\Sigma}) \cdot \mathcal{G}({}^\sigma\chi)^{\frac{n(n-1)}{2}} \cdot \Omega_{\chi_\infty^{(j)}\epsilon_\infty}({}^\sigma\Pi) \cdot \Omega_{\chi_\infty^{(j)}\epsilon_\infty}({}^\sigma\Sigma)}$$

- Generalize to more general Σ.

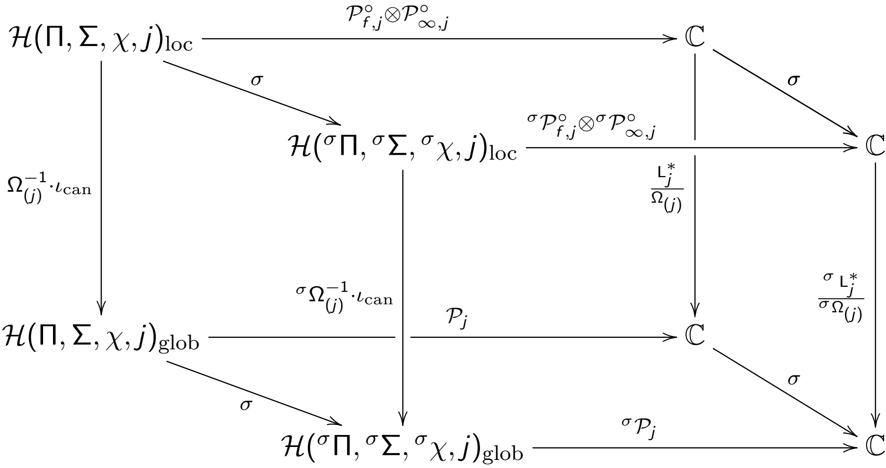
- When $n = 2$, this is proved by [Manin,Russian Math. Surveys 1976], [Shimura, CPAM 1976], [Shimura, Duke J. Math. 1978], [Harder, Progress in Mathematics, 1983], [Hida, Duke Math. J. 1994]
- Partial or conditional results for general n: Kazhdan-Mazur-Schmidt, Kasten-Schmidt, Mahnkopf, Raghuram, Raghuram-Shahidi, Grobner-Harris, Grobner-Raghuram, Harder-Raghuram, Januszewski, Grobner-Lin $\cdots$

Proof:

Modular symbol (algebraic topology).

Difficulty:

non-vanishing hypothesis and archimedean period relation.


$\mathcal{H}(\Pi,\Sigma,\chi,j)_{\mathrm{loc}}$
$\mathcal{P}^{\circ}_{f,j}\otimes\mathcal{P}^{\circ}_{\infty,j}$
$\mathbb{C}$
σ
σ
$\mathcal{H}({}^\sigma\Pi,{}^\sigma\Sigma,{}^\sigma\chi,j)_{\mathrm{loc}}$
${}^\sigma\mathcal{P}^{\circ}_{f,j}\otimes{}^\sigma\mathcal{P}^{\circ}_{\infty,j}$
$\mathbb{C}$
$\Omega_{(j)}^{-1}\cdot\iota_{\mathrm{can}}$
$\frac{\mathrm{L}_j^*}{\Omega_{(j)}}$
${}^\sigma\Omega_{(j)}^{-1}\cdot\iota_{\mathrm{can}}$
$\frac{{}^\sigma\,\mathrm{L}_j^*}{{}^\sigma\Omega_{(j)}}$
$\mathcal{H}(\Pi,\Sigma,\chi,j)_{\mathrm{glob}}$
$\mathcal{P}_j$
$\mathbb{C}$
σ
σ
$\mathcal{H}({}^\sigma\Pi,{}^\sigma\Sigma,{}^\sigma\chi,j)_{\mathrm{glob}}$
${}^\sigma\mathcal{P}_j$
$\mathbb{C}$

8. Standard L-functions of symplectic type

- $\mathrm{GL}_{2n}(\mathbb{A}_{\mathrm{k}}) \curvearrowright \Pi$: irreducible cuspidal automorphic, regular algebraic, symplectic type ($\mathrm{L}(s, \wedge^2 \otimes \eta)$ has a pole at 1), balanced coefficient system.
- $\chi : \mathrm{k}^\times \backslash \mathbb{A}_{\mathrm{k}}^\times \to \mathbb{C}^\times$: finite order character
- $\frac{1}{2} + j$: critical place.

Theorem (Jiang-Sun-Tian, preprint, 2019)

$$\frac{\mathrm{L}(\frac{1}{2} + j, \Pi \otimes \chi)}{\mathrm{i}^{jn \cdot [\mathrm{k} : \mathbb{Q}]} \cdot \mathcal{G}(\chi)^n \cdot \Omega_{\chi_\infty^{(j)}}(\Pi)} \in \mathbb{Q}(\Pi, \eta, \chi).$$

- Partial or conditional results: [Ash-Ginzburg, Invent. Math. 1994], [Grobner-Raghuram, Amer. J. of Math. 2014], [Januszewski, preprint 2018] $\cdots$

Thank you!

$$\mathcal{G}(\chi_{\mathbb{K}}) := \mathcal{G}(\chi_{\mathbb{K}}, \psi_{\mathbb{K}}, y_{\mathbb{K}}) := \int_{\mathcal{O}_{\mathbb{K}}^{\times}} \chi_{\mathbb{K}}(x)^{-1} \cdot \psi_{\mathbb{K}}(y_{\mathbb{K}} x) \, \mathrm{d}x,$$

$$\mathfrak{c}(\psi_{\mathbb{K}}) = y_{\mathbb{K}} \cdot \mathfrak{c}(\chi_{\mathbb{K}}).$$