Theta correspondence and special unipotent representations*

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Contents

1 Dual pairs and theta lifting

Basics of Howe's theory

- W: a finite-dimensional real symplectic vector space.
- σ: the anti-involution of $\mathrm{End}_{\mathbb{R}}(W)$ determined by $\langle\,,\,\rangle_W$.
- (A, A'): a pair of σ-stable semisimple $\mathbb{R}$-subalgebras of $\mathrm{End}_{\mathbb{R}}(W)$ that are mutual centralizers of each other.
- $G := A \cap \mathrm{Sp}(W)$ and $G' := A' \cap \mathrm{Sp}(W)$.
- (G, G'): a reductive dual pair in $\mathrm{Sp}(W)$.
  - irreducible if the algebra A (or A') is either simple or the product of two simple algebras that are exchanged by σ.

Construction of irreducible dual pairs:

- (D, σ_0): an $\mathbb{R}$-algebra with an anti-involution

$$(\mathbb{R}, \mathrm{id}), \quad (\mathbb{C}, \mathrm{id}), \quad (\mathbb{C}, {}^{-}), \quad (\mathbb{H}, {}^{-}),$$

$$(\mathbb{R}\times\mathbb{R}, \leftrightarrow), \quad (\mathbb{C}\times\mathbb{C}, \leftrightarrow), \quad (\mathbb{H}\times\mathbb{H}, (x, y)\mapsto(\bar{y}, \bar{x})).$$

- V: an ϵ-Hermitian right D-module, where $\epsilon = \pm 1$.
- $\mathrm{G}(V)$: the isometry group of V, is a classical Lie group:
  - a real orthogonal/symplectic group;
  - a complex orthogonal/symplectic group;
  - a unitary group;
  - a quaternionic symplectic/orthogonal group;
  - a real/complex/quaternionic general linear group.

- V': an ϵ'-Hermitian right D-module, where $\epsilon\epsilon' = -1$.
- $W := \mathrm{Hom}_{\mathrm{D}}(V, V')$, with the symplectic form

$$\langle T, S\rangle_W := \mathrm{Tr}_{\mathbb{R}}(T^* S), \qquad T,\, S \in \mathrm{Hom}_{\mathrm{D}}(V, V').$$

- Identify $\mathrm{G}(V)$ and $\mathrm{G}(V')$ with subgroups of $\mathrm{Sp}(W)$ via the natural homomorphism: $\mathrm{G}(V) \times \mathrm{G}(V') \longrightarrow \mathrm{Sp}(W)$:

$$(g, g') \cdot T = g' T g^{-1}, \qquad g \in G,\, g' \in G',\, T \in W.$$

- If both V and V' are nonzero, then $(\mathrm{G}(V), \mathrm{G}(V'))$ is an irreducible reductive dual pair in $\mathrm{Sp}(W)$.
- All irreducible reductive dual pairs arise in this way.

(G, G'): a reductive dual pair in $\mathrm{Sp}(W)$.

- $\mathrm{H}(W) := W \times \mathbb{R}$, the Heisenberg group with group multiplication

  $$(u,t)\cdot(u',t') = (u+u', t+t'+\langle u, u'\rangle_W), \qquad u, u' \in W,\ t, t' \in \mathbb{R}.$$

- Fix a nontrivial unitary character $\psi : \mathbb{R} \to \mathbb{C}^\times$.
- **Stone-von Neumann Theorem**: there exists a unique irreducible unitary representation of $\mathrm{H}(W)$ with central character ψ.

- Define the Jacobi group

$$J := (\widetilde{G} \times \widetilde{G}') \ltimes \mathrm{H}(W),$$

where $\widetilde{G}$ and $\widetilde{G}'$ are finite fold coverings of G and G'.

  - e.g. take the inverse image of (G, G') in $\widetilde{\mathrm{Sp}}(W)$ (the real metaplectic group).

- Assume that J has a unitary representation $\widehat{\omega}$ such that $\widehat{\omega}|_{\mathrm{H}(W)}$ is irreducible with central character ψ.
  - All such representations, if they exist, are isomorphic to each other up to twisting by unitary characters. Fix one $\widehat{\omega}$.
  - ω: the space of smooth vectors of $\widehat{\omega}|_{\mathrm{H}(W)}$.
    * called a <u>smooth oscillator representation</u> of J.

π: a Casselman-Wallach representation of $\widetilde{G}$.

- The <u>full theta lift</u> of π:

$$\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi) := (\omega \widehat{\otimes} \pi^\vee)_{\widetilde{G}}, \qquad \text{(the Hausdorff coinvariant space).}$$

- The <u>theta lift</u> $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ of π:

$$\text{the largest semisimple quotient of } \Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi).$$

- **Howe duality theorem**: if π is irreducible, then $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ is irreducible or zero. Consequently,
  - theta lifting is injective: for any irreducible π_1 and π_2 of $\widetilde{G}$, if $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi_1) \cong \theta_{\widetilde{G}}^{\widetilde{G}'}(\pi_2) \neq \{0\}$, then $\pi_1 \cong \pi_2$.

2 Theta lifting via matrix coefficient integrals

V: an ϵ-Hermitian right D-module. Fix a maximal compact subgroup K_V of $\mathrm{G}(V)$.

- Ψ_V: the function of $\mathrm{G}(V)$ such that
  - it is bi-K_V-invariant; and for all hyperbolic elements $g \in \mathrm{G}(V)$,

$$\Psi_V(g) = \prod_a \left(\frac{1+a}{2}\right)^{-\frac{1}{2}},$$

    where a runs over all eigenvalues of $g \otimes 1 : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$, counted with multiplicities.

- Ξ_V: the bi-K_V-invariant Harish-Chandra's Ξ function on $\mathrm{G}(V)$.

- $$\nu_V := \mathrm{rank}_{\mathrm{D}}(V) - \frac{2\dim_{\mathbb{R}}\{t \in \mathrm{D} \mid t^{\sigma_0} = \epsilon t\}}{\dim_{\mathbb{R}}(\mathrm{D})}.$$
- If $\mathrm{G}(V)$ is noncompact, then ν_V is the smallest real number such that
$$\Psi_V^{\nu_V} \cdot \Xi_V^{-1} \text{ is bounded.}$$

- Given $\nu \in \mathbb{R}$, a positive function Ψ on $\mathrm{G}(V)$ is said to be <u>ν-bounded</u> if there is a real number $r > 0$ such that

$$\Psi(kak') \leq (\log(3 + \mathrm{Tr}_{\mathbb{R}}(a)))^r \cdot \Psi_V^\nu(a) \cdot \Xi_V(a)$$

for all $k, k' \in K_V$ and all hyperbolic elements $a \in \mathrm{G}(V)$.

- A Casselman-Wallach representation π of $\widetilde{G}$ is said to be <u>ν-bounded</u> if there exist a ν-bounded positive function Ψ on $\mathrm{G}(V)$, and continuous seminorms $|\cdot|_\pi$ and $|\cdot|_{\pi^\vee}$ on π and $\pi^\vee$ (respectively) such that

$$|\langle \tilde{g} \cdot u, v\rangle| \leq \Psi(g) \cdot |u|_\pi \cdot |v|_{\pi^\vee}$$

for all $u \in \pi$, $v \in \pi^\vee$, and $\tilde{g} \in \widetilde{G}$.

Let π be a Casselman-Wallach representation of $\widetilde{G}$. Assume that π is genuine, namely the kernel of $\widetilde{G} \to G$ acts on π and ω by the same character.

- π is said to be convergent for $\Theta_{\widetilde{G}}^{\widetilde{G}'}$ if it is ν-bounded for some $\nu > \nu_V - \mathrm{rank}_{\mathrm{D}}(V')$.
- Then the integral

$$\begin{array}{rcl} \omega \times \pi^\vee \times \bar{\omega} \times \pi & \to & \mathbb{C}, \\ (\phi, v', \phi', v) & \mapsto & \int_G \langle \tilde{g} \cdot \phi, \phi' \rangle \cdot \langle \tilde{g} \cdot v', v \rangle \, dg, \end{array}$$

  is absolutely convergent and it yields a continuous bilinear map

$$(\omega \widehat{\otimes} \pi^\vee) \times (\bar{\omega} \widehat{\otimes} \pi) \to \mathbb{C}.$$

- Define

$$\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi) := \frac{\omega \widehat{\otimes} \pi^{\vee}}{\text{the left kernel of the bilinear map}}.$$

This is a quotient of $\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$, and hence a Casselman-Wallach representation of $\widetilde{G}'$.

3 Preservation of unitarity

- π is said to be overconvergent for $\Theta_{\widetilde{G}}^{\widetilde{G}'}$ if it is ν-bounded for some $\nu > \nu_V^\circ - \mathrm{rank}_{\mathrm{D}}(V')$, where

$$\nu_V^\circ := \begin{cases} \nu_V + 1, & \text{if } G \text{ is a real/complex odd orthogonal group;} \\ \nu_V + \frac{1}{2}, & \text{if } G \text{ is a quaternionic symp./orth. group;} \\ \nu_V, & \text{otherwise.} \end{cases}$$

- **Theorem**: Assume that $\mathrm{rank}_{\mathrm{D}}(V') \geq \nu_V^\circ$, and π is overconvergent for $\Theta_{\widetilde{G}}^{\widetilde{G}'}$. If π is unitarizable, so is $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$.

Remarks:

- Given that $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ is unitarizable, it is a semisimple quotient of $\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$. Thus if π is irreducible and $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi) \neq \{0\}$, then Howe Duality Theorem implies that $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi) = \bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ and is irreducible.

- Important earlier work in the same direction were due to Li and He.

4 Combinatorial parameters for special unipotent representations

- We illustrate using the examples of real even orthogonal groups and real symplectic groups, and
- construct a parameter set which underlies the special unipotent representations of both groups.

Notation: For a Young diagram $\imath$, write $\mathbf{R}_i(\imath)$ and $\mathbf{C}_i(\imath)$ $(i \in \mathbb{N}^+)$ respectively for its i-th row length and i-th column length.

- Let $\check{\mathcal{O}}$ be a nonempty Young diagram which satisfies the following good parity condition (for type D and C):

$$\text{All nonzero row lengths of } \check{\mathcal{O}} \text{ are } \underline{\text{odd}}.$$

$$m := |\check{\mathcal{O}}| := \sum_{i=1}^{\infty} \mathbf{R}_i(\check{\mathcal{O}}) \qquad \text{and} \qquad l := \mathbf{C}_1(\check{\mathcal{O}}).$$

- Define a pair $(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}})$ of Young diagrams such that the nonzero column lengths are given by

$$\begin{cases} \mathbf{C}_i(\imath_{\check{\mathcal{O}}}) = \frac{\mathbf{R}_{2i}(\check{\mathcal{O}})+1}{2}, \quad 1 \leq i \leq \frac{l-1}{2}; \\ \mathbf{C}_i(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{R}_{2i-1}(\check{\mathcal{O}})-1}{2}, \quad 1 \leq i \leq \frac{l+1}{2}, \end{cases}$$

if l is odd, and

$$\begin{cases} \mathbf{C}_i(\imath_{\check{\mathcal{O}}}) = \frac{\mathbf{R}_{2i-1}(\check{\mathcal{O}})+1}{2}, \quad 1 \leq i \leq \frac{l}{2}; \\ \mathbf{C}_i(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{R}_{2i}(\check{\mathcal{O}})-1}{2}, \quad 1 \leq i \leq \frac{l}{2}. \end{cases}$$

if l is even.

We introduce the set $\mathrm{BOX}(\imath)$ of boxes of a Young diagram $\imath$:

$$\mathrm{BOX}(\imath) := \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid j \leq \mathbf{R}_i(\imath)\},$$

and introduce five symbols $\bullet$, s, r, c and d to fill the boxes.

- A painting on a Young diagram $\imath$ is a map

  $$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

  with the following properties:

  - $\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;
  - when $S = \{s\}$ or $\{r\}$, every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$;
  - when $S = \{c\}$ or $\{d\}$, every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$.

- Define $\mathrm{PBP}(\check{\mathcal{O}})$ to be the set of all pairs $(\mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ and $\mathcal{Q}$ are paintings on $\imath_{\check{\mathcal{O}}}$ and $\jmath_{\check{\mathcal{O}}}$ respectively, such that
  - $\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$;
  - the image of $\mathcal{P}$ is contained in

$$\begin{cases} \{\bullet, r, c, d\}, & \text{if } l \text{ is odd;} \\ \{\bullet, s, r, c, d\}, & \text{if } l \text{ is even.} \end{cases}$$

  - the image of $\mathcal{Q}$ is contained in

$$\begin{cases} \{\bullet, s\}, & \text{if } l \text{ is odd;} \\ \{\bullet\}, & \text{if } l \text{ is even.} \end{cases}$$

- We call $(\mathcal{P}, \mathcal{Q})$ a painted bipartition attached to $\check{\mathcal{O}}$.

For $\tau = (\mathcal{P}, \mathcal{Q}) \in \mathrm{PBP}(\check{\mathcal{O}})$, we associate a classical group G_τ as follows.

- If l is odd, define $G_\tau := \mathrm{Sp}_{m-1}(\mathbb{R})$.
- If l is even, define the signature (p_τ, q_τ) by counting the various symbols appearing in $(\imath_{\check{\mathcal{O}}}, \mathcal{P})$, $(\jmath_{\check{\mathcal{O}}}, \mathcal{Q})$:

$$\begin{cases} p_\tau := (\#\bullet) + 2(\#r) + (\#c) + (\#d); \\ q_\tau := (\#\bullet) + 2(\#s) + (\#c) + (\#d). \end{cases}$$

Define $G_\tau := \mathrm{O}(p_\tau, q_\tau)$. Also define $\varepsilon_\tau \in \mathbb{Z}/2\mathbb{Z}$ such that $\varepsilon_\tau = 0$ if and only if the symbol d occurs in the first column of $\mathcal{P}$.

- If $l > 1$, we define $\check{\mathcal{O}}'$ to be the Young diagram obtained from $\check{\mathcal{O}}$ by removing the first row.
- There is a (combinatorially defined) descent map

$$\nabla : \mathrm{PBP}(\check{\mathcal{O}}) \to \mathrm{PBP}(\check{\mathcal{O}}').$$

- Define $\mathrm{PP}(\check{\mathcal{O}})$ to be the set of all $i \in \mathbb{N}^+$ such that

$$\mathbf{R}_i(\check{\mathcal{O}}) > \mathbf{R}_{i+1}(\check{\mathcal{O}}) > 0 \quad \text{and} \quad i \equiv l \pmod 2.$$

- Define the (extended) parameter set

$$\mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}}) := \mathrm{PBP}(\check{\mathcal{O}}) \times \{\wp \subset \mathrm{PP}(\check{\mathcal{O}})\}.$$

For each $(\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})$, we will construct a representation $\pi_{\tau,\wp}$ of G_τ.

5 Construction and classification

$\check{\mathcal{O}}$: a nonempty Young diagram satisfying the good parity condition, and $(\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})$.

- Let $G := G_\tau$, whose complexification $G_\mathbb{C}$ equals $\mathrm{Sp}_{m-1}(\mathbb{C})$ or $\mathrm{O}_m(\mathbb{C})$ respectively when l is odd or even.
- The Langlands dual of $G_\mathbb{C}$ is defined to be $\mathrm{O}_m(\mathbb{C})$.
- View $\check{\mathcal{O}}$ as a nilpotent $\mathrm{O}_m(\mathbb{C})$-orbit in $\mathfrak{o}_m(\mathbb{C})$.

- Take an $\mathfrak{sl}_2$-triple $(\check{e}, \check{h}, \check{f})$ in $\mathfrak{o}_m(\mathbb{C})$ such that $\check{e} \in \check{\mathcal{O}}$. Then $\frac{1}{2}\check{h}$ is a semisimple element of $\mathfrak{o}_m(\mathbb{C})$, which determines a character $\chi(\check{\mathcal{O}}) : \mathcal{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$ in the usual way.
- By a theorem of Dixmier, there exists a unique maximal G-stable ideal of $\mathcal{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.
- The associated variety of $I_{\check{\mathcal{O}}}$ is the closure of a nilpotent orbit $\mathcal{O} \in \mathrm{Nil}_{G_{\mathbb{C}}}(\mathfrak{g})$, called the Barbasch-Vogan dual of $\check{\mathcal{O}}$.

Definition: (Barbasch and Vogan) an irreducible Casselman-Wallach representation π of G is said to be special unipotent attached to $\check{\mathcal{O}}$ if $I_{\check{\mathcal{O}}}$ annihilates π.

Notation: $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the set of equivalent classes of irreducible Casselman-Wallach representations of G that are special unipotent attached to $\check{\mathcal{O}}$.

Put

$$\operatorname{Unip}(\check{\mathcal{O}}) := \begin{cases} \operatorname{Unip}_{\check{\mathcal{O}}}(\operatorname{Sp}_{m-1}(\mathbb{R})), & \text{if } l \text{ is odd;} \\ \bigsqcup_{p,q\in\mathbb{N},p+q=m} \operatorname{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } l \text{ is even.} \end{cases}$$

Theorem: Let $\check{\mathcal{O}}$ be a nonempty Young diagram which satisfies the good parity condition. Then

$$\#(\operatorname{Unip}(\check{\mathcal{O}})) = \begin{cases} \#(\mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})), & \text{if } l \text{ is odd;} \\ 2\#(\mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})), & \text{if } l \text{ is even.} \end{cases}$$

For each $(\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})$, we shall construct an irreducible Casselman-Wallach representation $\pi_{\tau,\wp}$ of G by induction on l.

- $l = 1$: the Young diagram $\check{\mathcal{O}}$ has only one row. Then $G = \mathrm{Sp}_{m-1}(\mathbb{R})$, and the set $\mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})$ has a unique element. We define $\pi_{\tau,\wp}$ to be the trivial representation of G.

- $l \geq 2$: write $\tau' := \nabla(\tau) \in \mathrm{PBP}(\check{\mathcal{O}}')$, and define

$$\wp' := \{i \in \mathbb{N}^+ \mid i + 1 \in \wp\} \subset \mathrm{PP}(\check{\mathcal{O}}').$$

  Write $m' := |\check{\mathcal{O}}'|$ and $G' := G_{\tau'}$.

- G and G' form a reductive dual pair in $\mathrm{Sp}(W)$, where W is a real symplectic space of dimension $(m-1)m'$ or $m(m'-1)$, respectively when l is odd or even.

- Let $J = (G \times G') \ltimes \mathrm{H}(W)$ and ω be a smooth oscillator representation (in which the orthogonal group acts via the natural linear action in a Schrodinger model).
- By induction hypothesis, we have an irreducible Casselman-Wallach representation $\pi_{\tau',\wp'}$ of G'. Define

$$\pi_{\tau,\wp} := \begin{cases} \Theta_{G'}^{G}(\pi_{\tau',\wp'}^{\vee} \otimes \det^{\varepsilon_\wp}), & \text{if } l \text{ is odd;} \\ \Theta_{G'}^{G}(\pi_{\tau',\wp'}^{\vee}) \otimes (1_{p_\tau,q_\tau}^{+,-})^{\varepsilon_\tau}, & \text{if } l \text{ is even.} \end{cases}$$

Here $\varepsilon_\wp$ denote the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\wp = 1 \Leftrightarrow 1 \in \wp.$$

Theorem:

(a) For every $(\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})$, the representation $\pi_{\tau,\wp}$ of G_τ is irreducible, unitarizable, and special unipotent attached to $\check{\mathcal{O}}$.

(b) Suppose that l is odd so that $G = \mathrm{Sp}_{m-1}(\mathbb{R})$. Then the following map is bijective:

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}}) & \rightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ (\tau, \wp) & \mapsto & \pi_{\tau,\wp}. \end{array}$$

(c) Suppose that l is even, and p, q are non-negative integers with $p + q = m$. Then the following map is bijective:

$$\begin{array}{rcl} \left\{ \begin{array}{l} (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}}) \mid \\ (p_\tau, q_\tau) = (p, q) \end{array} \right\} \times \mathbb{Z}/2\mathbb{Z} & \rightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), \\ ((\tau, \wp), \epsilon) & \mapsto & \pi_{\tau,\wp} \otimes \det{}^{\epsilon}. \end{array}$$

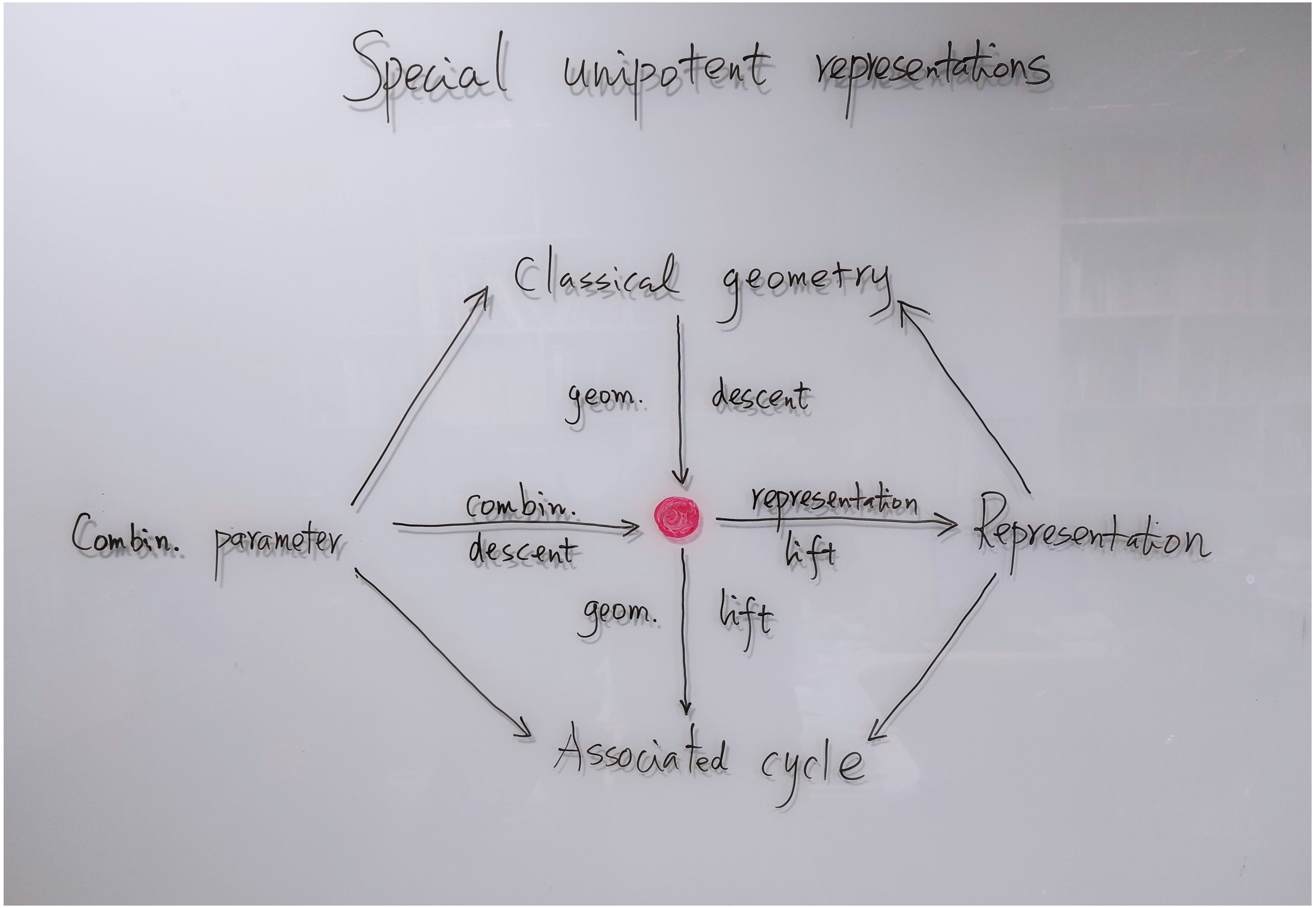
- We have thus explicitly constructed all special unipotent representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$, when all row lengths of $\check{\mathcal{O}}$ are odd.
- If some row lengths of $\check{\mathcal{O}}$ are even, then they must come in pairs. Via irreducible unitary parabolic inductions, the construction of representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is reduced to the case when all row lengths of $\check{\mathcal{O}}$ are odd.
- In the same approach, we may parameterize and construct all special unipotent representations of the real classical groups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{U}(p,q)$, $\mathrm{O}(p,q)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{O}^*(2n)$, $\mathrm{Sp}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, as well as all metaplectic special unipotent representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$.

Theorem: (confirming the Arthur-Barbasch-Vogan conjecture for real classical groups)

- All special unipotent representations of the real classical groups are unitarizable;
- all metaplectic special unipotent representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ are also unitarizable.

Remark: The unitarizability of special unipotent representations for quasisplit classical groups is independently due to Adams, Arancibia Robert and Mezo, as a consequence of their result

$$\text{Arthur packet} = \text{ABV packet}.$$



A schematic diagram

Thank you!