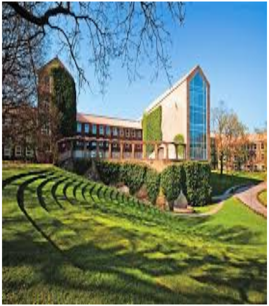
Twisted GGP Problems and Conjectures

(Joint with Benedict Gross and Dipendra Prasad)



Basic Fourier-Jacobi Case of GGP

- E/F quadratic extension of local fields, with quadratic character $\omega_{E/F}$ and $\mathrm{Aut}(E/F) = \langle \sigma \rangle$.
- V a skew-Hermitian space over E, isometry group $\mathrm{U}(V)$
- $\omega_{\psi,\mu,V}$ a Weil representation of $\mathrm{U}(V)$: depends on a character μ of $E^\times$ with $\mu|_{F^\times} = \omega_{E/F}$ and a nontrivial character ψ of F.

For $\pi_1, \pi_2 \in \mathrm{Irr}(\mathrm{U}(V))$,

$$\mathrm{Hom}_{\mathrm{U}(V)}(\pi_1 \otimes \pi_2, \omega_{\psi,\mu,V}).$$

Its dimension is ≤ 1 (B.Y. Sun), and the GGP conjecture gives a precise criterion for its nonvanishing (in terms of LLC). This was proved by Ichino and myself in the nonarchimedean case.

A Twisted Version

Instead of

$$\mathrm{U}(V) = \mathrm{U}(V)(F) \hookrightarrow \mathrm{U}(V) \times \mathrm{U}(V) = \mathrm{U}(V)(F \times F).$$

one can replace $F \times F$ by any quadratic field K/F, so that

$$\mathrm{U}(V) = \mathrm{U}(V)(F) \hookrightarrow \mathrm{U}(V_K) = \mathrm{U}(V)(K),$$

where V_K is a skew-Hermitian space relative to $E \otimes_F K/K$.

The twisted GGP problem considers

$$\mathrm{Hom}_{\mathrm{U}(V)}(\Pi, \omega_{\psi,\mu,V}) \quad \text{for } \Pi \in \mathrm{Irr}(\mathrm{U}(V_K)).$$

Remark: Symplectic case?

Special Case: $E = K$

In general, (E, K) can be any pair of étale quadratic F-algebras.

$E \backslash K$	$F \times F$	E	field
$F \times F$	Rankin-Selberg	Rankin-Selberg	Asai
field	GGP	$\mathrm{U}(V) \subset \mathrm{GL}(V)$	$\mathrm{U}(V) \subset \mathrm{U}(V_K)$

We first examine the special case when $K = E$, so that $E \otimes_F K \cong E \times E$ and

$$\mathrm{U}(V) \hookrightarrow \mathrm{U}(V_K) \cong \mathrm{GL}(V).$$

This case might be simpler than the original GGP: the L-packets of $\mathrm{GL}(V)$ are singletons.

Skew-Hermitian spaces

Over local archimedean local fields, skew-Hermitian spaces V of a fixed dimension n are classified by their determinants:

$$\det(V) \in \begin{cases} F^\times / N_{E/F}(E^\times), & \text{if } n \text{ is even;} \\ E_0^\times / N_{E/F}(E^\times), & \text{if } n \text{ is odd} \end{cases}$$

where $E_0^\times$ is the set of nonzero trace 0 elements of E. There are thus 2 skew-Hermitian spaces of dim. n.

For $\mathbb{C}/\mathbb{R}$, V is classified by its signature

$$(p, q) \longleftrightarrow p \text{ eigenvalues } i\text{'s and } q \text{ eigenvalues } -i\text{'s}$$

with $p + q = n$.

Local Conjecture: $E = K$

(i) For any $\Pi \in \mathrm{Irr}(\mathsf{GL}(V))$, $\dim \mathrm{Hom}_{\mathrm{U}(V)}(\Pi, \omega_{V,\psi,\mu}) \leq 1$.

(ii) For $\Pi \in \mathrm{Irr}(\mathsf{GL}(V))$ generic,

$$\sum_V \dim \mathrm{Hom}_{\mathrm{U}(V)}(\Pi, \omega_{V,\psi,\mu}) = 1.$$

where the sum is over skew-hermitian structures on V.

(iii) For generic $\Pi \in \mathrm{Irr}(\mathsf{GL}(V))$, the unique V which gives a nonzero contribution to the above sum satisfies:

$$\mu(\det(V)) = \epsilon(\frac{1}{2}, \Pi \times {}^\sigma\Pi^\vee \times \mu^{-1}, \psi_E) \cdot \omega_\Pi(-1)^n \cdot \omega_{E/F}(-1)^{n(n-1)/2},$$

where ${}^\sigma\Pi^\vee$ is the conjugate-dual representation of Π and ω_Π is the central character of Π.

Remarks

- Observe that the conjecture does not require the LLC for its formulation, so in a sense, it is simpler than GGP.
- If M is the L-parameter of Π, then $M \otimes {}^{\sigma}M^{\vee}$ is conjugate-orthogonal.
- (iii) can be written as:

$$\mu(\det(V)) =$$

$$\epsilon(\frac{1}{2}, M \otimes {}^{\sigma}M^{\vee} \times \mu^{-1}, \psi_E) \cdot \det(M)(-1)^n \cdot \omega_{E/F}(-1)^{n(n-1)/2}.$$

Let $e \in E_0^{\times}$. Then the two signs on the RHS can be rephrased using:

$$\det(M)(-1)^n = \det(M \otimes {}^{\sigma}M^{\vee})(e)$$

and

$$\omega_{E/F}(-1) = \omega_{K/F}(e^2).$$

This is the form which generalizes well.

Evidences

- When $n = \dim V = 1$, the conjecture reduces to the theorem of Moen and Rogawski on decomposition of the Weil representation of U_1.
- When $n = 2$, it reduces to the theorem of trilinear forms, due to Prasad, for

$$\mathrm{GL}_2(F) \text{ or } B^\times \hookrightarrow \mathrm{GL}_2(E)$$

- We can prove the conjecture for unitary principal series representations.
- Indeed, the conjecture can be reduced to the case of discrete series representations.

Reduction to Discrete Series

The reduction is based on the following hereditary property:

Proposition

Let $\Pi = \Pi_1 \times \Pi_2$ be a tempered rep. of $GL_n(E)$ parabolically induced from a maximal parabolic with Levi $GL_{n_1} \times GL_{n_2}$, with Π_i tempered rep. on GL_{n_i}. Then

$$\mathrm{Hom}_{\mathrm{U}(V)}(\Pi, \omega_{\psi,\mu,V}) \cong$$

$$\bigoplus_{(V_1,V_2)} \mathrm{Hom}_{\mathrm{U}(V_1)}(\Pi_1, \omega_{\psi,\mu,V_1}) \otimes \mathrm{Hom}_{\mathrm{U}(V_2)}(\Pi_2, \omega_{\psi,\mu,V_2})$$

where the sum is over all pairs (V_1, V_2) of skew-Hermitian spaces of dim. n_1 and n_2 with $V_1 + V_2 \cong V$.

Using this inductively, one reduces the principal series case to the case $n = 1$. Since we know the conjecture for $n \leq 2$, we also know it for tempered reps with square-integrable support $\mathrm{GL}_1^a \times \mathrm{GL}_2^b$.

Conjecture in Archimedean Case

Assume that $E/F = \mathbb{C}/\mathbb{R}$. Let

$$\Pi = \mathrm{Ind}_B^{\mathrm{GL}_n(\mathbb{C})}(\chi_1 \otimes \cdots \otimes \chi_n)$$

with

$$\chi_j(z) = |z|^{r_j} \cdot (\bar{z}/z)^{k_j/2}, \quad k_j \in \mathbb{Z},$$

and let

$$\mu(z) = \left(\frac{\bar{z}}{z}\right)^{\alpha} \quad \text{with } \alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}.$$

With $\psi(x) = e^{2\pi i x}$,

$$\mathrm{Hom}_{\mathrm{U}(V_{p,q})}(\Pi, \omega_{V_{p,q},\psi,\mu}) \neq 0$$

if and only if

$$\#\{j : k_j > \alpha\} = p \quad \text{and} \quad \#\{j : k_j < \alpha\} = q = n - p.$$

Global Conjectures: $E = K$

Now let E/F be global fields and $\Pi = \otimes_v \Pi_v$ be a cuspidal representation of $\mathrm{GL}(V)$. Have global period integral:

$$\mathcal{P} : \Pi \otimes \overline{\omega_{V,\psi,\mu}} \longrightarrow \mathbb{C}$$

defined by

$$\mathcal{P}_V(f, \phi) = \int_{[\mathrm{U}(V)]} f(g) \cdot \overline{\phi(g)}\, dg$$

Conjecture: $\mathcal{P}_V \neq 0$ if and only if

(a) For all places v of F, $\mathrm{Hom}_{\mathrm{U}(V_v)}(\Pi_v, \omega_{V_v,\psi_v,\mu_v}) \neq 0$.

(b) $L(1/2, \Pi \times {}^{\sigma}\Pi^{\vee} \times \mu^{-1}) \neq 0$.

Further, if $L(1/2, \Pi \times {}^{\sigma}\Pi^{\vee} \times \mu^{-1}) \neq 0$, there exists a skew-hermitian space V over E such that $\mathcal{P}_V$ is nonzero.

Refined Global Conjecture

We also make a refined global conjecture in the style of Ichino-Ikeda. For each place v of F, consider the integral of matrix coefficients:

$$\mathcal{I}_v(f_v, f'_v, \phi_v, \phi'_v) := \int_{\mathrm{U}(V)(F_v)} \langle g_v \cdot f_v, f'_v \rangle \cdot \overline{\langle g_v \cdot \phi_v, \phi'_v \rangle} \, dg_v,$$

for $f_v, f'_v \in \Pi_v$ and $\phi_v, \phi'_v \in \omega_{\psi_v, \mu_v, V_v}$. This is absolutely convergent for tempered Π_v.

When all data is unramified, one conjectures that

$$\mathcal{I}_v(f_v, f'_v, \phi_v, \phi'_v) = \frac{L(1, M^\vee_{\mathrm{GL}(V_v)})}{L(1, M^\vee_{\mathrm{U}(V_v)})} \cdot \frac{L(1/2, \Pi_v \times {}^\sigma\Pi^\vee_v \times \mu_v^{-1})}{L(1, \Pi, \mathrm{Ad})},$$

where $L(1, M^\vee_{\mathrm{GL}(V_v)})$ and $L(1, M^\vee_{\mathrm{U}(V_v)})$ are the value at $s = 1$ of the L-function of the dual motive of $\mathrm{GL}(V)$ and $\mathrm{U}(V)$.

One then defines a normalized local period integral:

$$\mathcal{I}_v^{\#} = \frac{L(1, M^{\vee}_{\mathrm{U}(V_v)})}{L(1, M^{\vee}_{\mathrm{GL}(V_v)})} \cdot \frac{L(1, \Pi_v, \mathrm{Ad})}{L(1/2, \Pi_v \times {}^{\sigma}\Pi_v^{\vee} \times \mu_v^{-1})} \cdot \mathcal{I}_v.$$

Fixing decompositions

$$dg = \prod_v dg_v \quad \text{and} \quad \langle -, - \rangle_{\mathrm{Pet}} = \prod_v \langle -, - \rangle_v,$$

and using these local factors in the definition of the local period integrals $\mathcal{I}_v$, one conjectures:

Given a (tempered) cuspidal automorphic representation Π of $\mathrm{GL}(V)$,

$$\mathcal{P} \otimes \overline{\mathcal{P}} = \frac{L(1/2, \Pi \times {}^{\sigma}\Pi^{\vee} \times \mu^{-1})}{L(1, M^{\vee}_{\mathrm{U}(V)})} \cdot \left(\frac{L(s, M^{\vee}_{\mathrm{GL}(V)})}{L(s, \Pi, \mathrm{Ad})} \right) |_{s=1} \cdot \prod_v \mathcal{I}_v^{\#}.$$

Here, note that $L(s, M^{\vee}_{\mathrm{GL}(V)})$ and $L(s, \Pi, \mathrm{Ad})$ both have a simple pole at $s = 1$, so that their ratio is holomoprhic and nonzero at $s = 1$.

Some Ongoing Work

- Jialiang Zou is working on a Gelfand-Kazhdan type argument to show the multiplicity-at-most-one statement. The setup used by Sun for GGP does not extend to the twisted case.
- Chuijia Wang is working on showing the local conjecture for supercuspidal representations of $\mathrm{GL}(V)$, using their realizations as compactly induced representations. In his thesis, he has shown cases of Prasad's Galois period conjecture for regular supercuspidal representations, using a refined Mackey theory formula of Hakim and Murnaghan.
- Danielle Wang (a student of W. Zhang) is working on an RTF approach to the global conjecture. She has also verified the expectation for the local integral of unramified matrix coefficients for $n \leq 4$.

Preparation for General Case

Now assume $E \neq K$ are two quadratic field extensions of F.

$$\begin{tikzcd}
 & L = E \otimes K \arrow[dl, dash, "\sigma"'] \arrow[dr, dash, "\tau"] & \\
K \arrow[dr, dash, "\tau"'] & & E \arrow[dl, dash, "\sigma"] \\
 & F &
\end{tikzcd}$$

Lemma

The two skew-hermitian spaces $V_K = V \otimes_F K$ and $V'_K = V' \otimes_F K$ are isomorphic over L. Moreover, $\mathrm{U}(V_K) \cong \mathrm{U}(V'_K)$ is quasi-split.

Hence we may regard $\mathrm{U}(V)$ and $\mathrm{U}(V')$ as subgroups of a fixed $\mathrm{U}(V_K) = \mathrm{U}(V'_K)$.

LLC for $\mathrm{U}(V_K)$

$$\mathrm{Irr}(\mathrm{U}(V_K)) \longleftrightarrow \Phi(\mathrm{U}(V_K)) = \{(M, \eta)\}$$

For $\Pi \in \mathrm{Irr}(\mathrm{U}(V_K))$, its L-parameter is an n-dimensional conjugate-dual representation M of WD_L (relative to L/K) of sign $(-1)^{n-1}$. We may decompose:

$$M = \oplus_{i \in I} V_i \otimes M_i \oplus P \oplus {}^{\sigma}P^{\vee}$$

with M_i of same type as M, V_i its multiplicity space, and all irred. summands of P are not of same type as M. Then have the component group

$$A_M = \prod_{i \in I} \mathbb{Z}/2\mathbb{Z} \cdot a_i,$$

and Π corresponds to an element η of $\mathrm{Irr}(A_M)$ trivial on -1_M. Let

$$\Pi_M \longleftrightarrow \mathrm{Irr}(A_M/\langle -1_M \rangle)$$

be the L-packet of M.

Asai Factors

If M is an L-parameter for $\mathrm{U}(V_K)$, $M \otimes M^\tau$ is τ-invariant and so

$$\mathrm{Ind}_{WD_L}^{WD_E}(M \otimes M^\tau) = \mathrm{As}^+_{L/E}(M) \oplus \mathrm{As}^-_{L/E}(M).$$

(a) If $M = \oplus_i M_i$, then

$$\mathrm{As}^\epsilon_{L/E}(M) = \bigoplus_i \mathrm{As}^\epsilon_{L/E}(M_i) \oplus \bigoplus_{i<j} \mathrm{Ind}_L^E(M_i \otimes M_j^\tau).$$

(b) $\mathrm{As}^\epsilon_{L/E}(M)^\vee \cong \mathrm{As}^\epsilon_{L/E}(M^\vee)$.

(c) $\mathrm{As}^\epsilon_{L/E}(M_1 \otimes M_2) \cong \mathrm{As}^\epsilon_{L/E}(M_1) \otimes \mathrm{As}^\epsilon_{L/E}(M_2)$.

(d) If $\dim M = 1$, then $\mathrm{As}^+_{L/E}(M)$ is the restriction of M to $E^\times$.

(e) $\det(\mathrm{As}^+_{L/E}(M)) = \det(M)|^n_{E^\times} \cdot \omega_{L/E}^{n(n-1)/2}$.

(f) $\mathrm{As}^\pm_{L/E}(M)$ is conjugate-orthogonal relative to E/F.

Local Conjectures: $E \neq K$

(i) For each $\pi \in \mathrm{Irr}(\mathrm{U}(V_K))$,

$$m_V(\pi, \mu) = \dim \mathrm{Hom}_{\mathrm{U}(V)}(\pi, \omega_{V,\psi,\mu}) \leq 1.$$

(ii) Let M be a generic L-parameter of $\mathrm{U}(V_K)$ with associated L-packet $\Pi_M \subset \mathrm{Irr}(\mathrm{U}(V_K))$. Then

$$\sum_{V} \sum_{\pi \in \Pi_M} m_V(\pi, \mu) = 1$$

where the first sum runs over the two skew-hermitian spaces over E of dimension n and the second runs over the L-packet Π_M.

Local Conjectures Continued

(iii) The unique V_0 which contributes to (ii) is characterized by

$$\mu(\det(V_0)) =$$

$$\epsilon(1/2, \mathrm{As}^+_{L/E}(M) \otimes \mu^{-1}, \psi_E) \cdot \det(\mathrm{As}^+_{L/E}(M))(e) \cdot \omega_{K/F}(e^2)^{n(n-1)/2}$$

where $e \in E_0^\times$, so that $E = F(e)$.

(iv) The unique $\pi \in \Pi_M$ which contributes to (ii) corresponds to the following character of $A_M = \prod_{i \in I} \mathbb{Z}/2\mathbb{Z} \cdot a_i$:

$$\begin{aligned} \chi(a_i) &= \epsilon(1/2, \mathrm{Ind}_L^E(M_i^\tau \otimes (M/M_i)) \cdot \mu^{-1}, \psi_{E,e}) \\ &= \epsilon(1/2, [\mathrm{As}(M_i) + \mathrm{As}(M) + \mathrm{As}(M/M_i)] \cdot \mu^{-1}, \psi_{E,e}), \end{aligned}$$

where $\psi_{E,e}$ is the additive character of E/F defined by $\psi_{E,e}(x) = \psi(Tr(ex))$.

Evidences

- when $n = \dim V = 1$, this reduces to Moen-Rogawski.
- when $n = 2$, this reduces to a refinement of Prasad's trilinear form theorem:

  $$\mathrm{GL}_2(F) \text{ or } B^\times \hookrightarrow \mathrm{GL}_2(K)^+$$

  where

  $$\mathrm{GL}_2(K)^+ = \{g \in \mathrm{GL}_2(K) : \det(g) \in NL^\times\}.$$

  This refinement can be shown via theta correspondence.
- We can show the conjecture if M corresponds to unitary principal series, i.e.

  $$M = \bigoplus_i (\chi_i + {}^\sigma\chi_i^{-1}).$$

Some Discrete Series Cases

In ongoing joint work with Rui Chen, we are trying to prove the conjecture for discrete series parameters of the form

$$M = \chi_1 + \dots + \chi_n,$$

by induction on n. Writing $M = \chi + N$, the packet Π_M can be constructed as theta lifts from $\mathrm{U}(W_K) = \mathrm{U}_{n-1}$.

$$\begin{tikzcd}
\mathrm{U}(\mathrm{Res}_{L/E}(W_K)) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(V_K) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(W) & \mathrm{U}(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{U}(\mathrm{Res}_{L/E}(W_K)) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(V'_K) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(W) & \mathrm{U}(V')
\end{tikzcd}$$

Now, $\omega_{\psi,\mu,V}$ has A-parameter of shape $1 + S_{n-1}$. On $\mathrm{U}(\mathrm{Res}_{L/E}(W_K))$, one has $\Theta(\omega_{\psi,\mu,V})$, with A-parameter

$$1 + S_{n-1} + S_{n-2}.$$

Problems

- Can one show a geometric multiplicity formula for the branching multiplicity in the style of Waldspurger, Beuzart-Plessis, C. Wan and prove the conjecture via twisted endoscopic transfer to GL?
- RTF for global conjectures? Danielle Wang (student of W. Zhang) is working on the $E = K$ case.
- Finite Fields: one has

$$\mathrm{U}_n(F) \subset \mathrm{GL}_n(F) \quad \text{or} \quad \mathrm{Sp}_{2n}(F) \subset \mathrm{Sp}_{2n}(E)$$

  and can consider branching to the Weil representation of $\mathrm{U}_n(F)$ and $\mathrm{Sp}_{2n}(F)$.
- Arithmetic case: is there an arithmetic story analogous to that developed by Y.F. Liu for the basic unitary FJ case of GGP? Is there a new AFL?

Thank You for Your Attention