Induced representation of Hermitian Lie groups from Heisenberg parabolic subgroups

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Outline

- Questions. Results. Related/Earlier Results. Methods.
- Heisenberg parabolic subgroup P. Induced representation.
- Composition series, complementary series, unitary subrepresentations.
- Outlook: Further questions

Hermitian Lie group

- G, irreducible Hermitian Lie group. $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$, Cartan deco..
- $Z \in \mathfrak{k}$ center element. $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+$, HC deco..
- $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$, Cartan subalgebra. $\gamma_1, \cdots \gamma_r$, HC strongly orthogonal roots.
- e^+, e^- root vectors for $\gamma := \gamma_1$, $\{e^+, e^-, h = [e^+, e^-]\}$, sl_2 triple.
- $\xi := \xi_e = e^+ + e^- \in \mathfrak{p}$, $\mathfrak{a} = \mathbb{R}\xi \subset \mathfrak{p}$ one-dimensional Abelian subspace.

Heisenberg parabolic subgroup and Induced Representation

- $\mathfrak{g} = \mathfrak{n}_{-2} + \mathfrak{n}_{-1} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2$, root space decomposition of $\mathfrak{a} = \mathbb{R}\xi$.
- $P = MAN$, parabolic subgroup.
- $\nu \in \mathbb{C}$ viewed as $\xi_e \to \nu$.
- $I(\nu) := Ind_P^G(1 \otimes e^\nu \otimes 1)$, induced representation.

Results. Method

• Results: Composition series, complementary series and unitary subrepresentations of $I(\nu)$.

• Method: Computing recursions of multiplications and differentiations of spherical polynomials for line bundles over K/L_0 (projectivization of $G/P = K/L$).

Earlier related results and methods

- Kostant, Johnson-Wallach, Cowling-Koranyi, ...:
- Howe-Tan: $G = U(p, q)$, P Heisenberg parabolic; $G = SO(p, q)$.
- Knapp-Speh: $G = SU(2, n)$, P Heisenberg parabolic.
- Sahi: G conformal group of Jordan algebras (including Hermitian Lie algebra), P Siegel parabolic ($\neq$ Heisenberg parabolic.)
- Barbasch studied extensively the spherical dual of classical groups, e.g., $Sp(n, \mathbb{R})$, $Sp(n, \mathbb{C})$.

- S.-T. Lee, Lee-Loke, Lee-Zhu: Complex classical groups, $SO(p, q)$, $Sp(p, q)$..; in relation with dual pair correspondence.
- Johnson: $SO(n, n)$, $Sp(p, p)$.
- Many other works, ...
- Ørsted-Zhang, Zhang: G conformal group of Jordan algebras; we'll use similar method here.

Remark

- The appearance of Heisenberg groups in general simple Lie groups is classified by R. Howe (2008).
- A. Kaplan et al classified H-type groups in simple Lie groups (2018).

$\mathfrak{g}$-actions on $L^2(K/L)$

- Decompose $L^2(K/L)$.
- Find L-invariant elements $\{\phi\}$ in $L^2(K/L)$.
- Compute the action of $\xi = e^+ + e^-$ on $L^2(K/L)^L$ using recursion formulas for multiplication and differentiations of $\{\phi\}$.

Circle bundle $G/P = K/L \to K/L_0$ over compact Hermitian symmetric space K/L_0

Realize $D = G/K \subset V = \mathbb{C}^d$, bounded symmetric domain.

Induced representation realized on $L^2(S)$, $S = K/L$,

$$L = \{k \in K; ke = e\}, \quad \mathfrak{l} = \mathfrak{m} \cap \mathfrak{k} = \{X \in \mathfrak{k}; Xe = 0\} \subset \mathfrak{k}.$$

$S = K/L$ manifold of rank one tripotents in V (in the sense of Jordan triple product).

$$S_1 := \mathbb{P}(K/L) = \text{projectivization of } K/L = K/L_0 \subset \mathbb{P}(V),$$

$$L_0 = \{k \in K; ke = \chi(k)e\},$$

$\chi(k)$ defines a character of L_0. K/L_0 is a compact Hermitian symmetric space.

Example

$G/K = SU(r, r+b)$, $S = K/L$ = manifold of rank one partial isometries. $K/L_0 = \mathbb{P}^{r-1} \times \mathbb{P}^{r+b-1}$, projective space of rank one

$r \times (r+1)$-matrices.

$$\xi = \begin{bmatrix} & E_{11} \\ E_{11} & \end{bmatrix} \in \mathfrak{su}(r, r+b)$$

Tangent space $\mathfrak{q}$ of K/L_0 at $e = E_{11} \in K/L_0$, $\mathfrak{k} = \mathfrak{l}_0 + \mathfrak{q}$:

$$\mathfrak{q} = \{\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}\} \subset M_{r,r+b}$$

Example

- $G/K = E_{6(-14)}/Spin(10) \times SO(2)$, type V (exceptional) domain in $\mathbb{O}_{\mathbb{C}}^2 = \mathbb{C}^{16}$.

$K/L_0 = Spin(10) \times SO(2)/U(5) \times \times SO(2)$, compact dual of the classical domain $SO^*(10)/U(5)$ of 5×5-skew symmetric matrices.

(A complete table of (G, K) and (K, L_0) is given in the end.)

L^2-sections of line bundles over K/L_0

(χ^l, L_0) defines holomorphic line bundle $K \times_{(L_0,\chi^l)} \mathbb{C}$ over K/L_0.

Circle bundle

$$K/L \to K/L_0, \quad \chi : L \subset L_0 \to U(1), L = \operatorname{Ker} \chi.$$

$$L^2(K/L) = \sum_{l=-\infty}^{\infty} L^2(K/L_0, \chi_l),$$

$L^2(K/L_0, \chi_l) = \{L^2\}$-sections.

To find $L^2(K/L_0, \chi_l)$-deco.: Rank and HC roots for $(\mathfrak{k}, \mathfrak{l})$

- $\mathrm{rank}(K/L_0) = 1$ if $G = SU(n,1)$ or $G = Sp(n, \mathbb{R})$;

$\mathrm{rank}(K/L_0) = 2$ otherwise.

- $K/L_0 = K_1/L_1$ semisimple compact symmetric space

- α_1, α_2, HC strongly orthogonal roots for $(\mathfrak{k}_1, \mathfrak{l}_1)$, the

semisimple part of $\mathfrak{k}_1$.

- α_0, dual element of the center element Z_0.

L^2-decomposition

$$L^2(K/L_0, \chi_l) = \sum_{\mathbf{m}} W_{\mathbf{m},l},$$

$W_{\mathbf{m},l}$ of highest weight

$$l\alpha_0 + \mathbf{m}, \mathbf{m} = m_1\alpha_1 + m_2\alpha_2,$$

$$m_1 \geq m_2 \geq |l|, \quad m_1 = m_2 = l, \mathrm{mod}\, 2.$$

(l, l)-spherical invariants of L_0,

$$(W_{\mathbf{m},l})^{L_0,(l,l)} = \mathbb{C}\phi_{\mathbf{m},l}.$$

(Cartan-Helgason thm., generalized by Schlichtkrull)

$\mathfrak{g}$-action

L-invariant element $\xi = e^+ + e^- \in \mathfrak{g}$ acts on

$$\sum_{\mathbf{m}=(m_1,m_2),l} \mathbb{C}\phi_{\mathbf{m},l}.$$

Let $\mathfrak{g} \neq \mathfrak{su}(d,1), \mathfrak{sp}(r,\mathbb{R})$ and

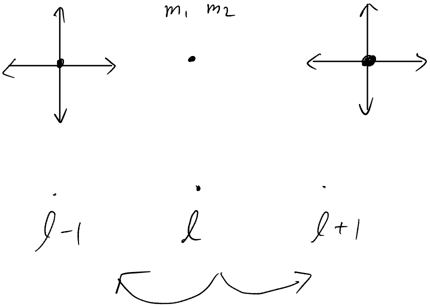
$\rho =$ half sum of positive roots for the symmetric pair $(\mathfrak{k}_1^*, \mathfrak{l}_1)$

$$\rho = \rho_1\alpha_1 + \rho_2\alpha_2.$$

Theorem

$$
\begin{aligned}
&\pi_\nu(\xi)\phi_{\mathbf{m},l} \\
&= \sum_{\sigma=(\sigma_1,\sigma_2)=(\pm1,\pm1)} \Big(\nu + \sigma_1(m_1+\rho_1) + \sigma_2(m_2+\rho_2) - (\rho_1+\rho_2)\Big) \\
&\quad \times \Big(c_{\mathbf{m},l}(\mathbf{m}+\sigma, l+1)\phi_{\mathbf{m}+\sigma,l+1} + c_{\mathbf{m},l}(\mathbf{m}+\sigma, l-1)\phi_{\mathbf{m}+\sigma,l-1}\Big).
\end{aligned}
$$

The coefficients can be expressed as quotients of HC C-functions.


m_1 m_2
$l-1$
l
$l+1$

Complementary series

Let $\rho_{\mathfrak{g}} =$ half sum of roots of $\mathfrak{g}$ with respect to $\mathfrak{a} = \mathbb{R}\xi$.

Theorem

The complementary series $I(\nu)$ appears precisely in the range $\nu = \rho_{\mathfrak{g}} + \delta$, $|\delta| < \delta_0$,

$$\delta_0 = \begin{cases} 1+b, & \mathfrak{g} = \mathfrak{su}(r+b, r) \\ 3, & \mathfrak{g} = \mathfrak{so}^*(2r) \\ n-3, & \mathfrak{g} = \mathfrak{so}(2, n), \quad n > 4 \\ 3, & \mathfrak{g} = \mathfrak{e}_{6(-14)} \\ 5, & \mathfrak{g} = \mathfrak{e}_{7(-25)} \end{cases}.$$

Theorem

Suppose $\nu \le 2\rho_2 - 2$ is an even integer. Then there are two unitarizable subrepresentations $S^{\pm}(\nu) \subset I(\nu)$ consisting of the K-types

$$S^+(\nu) = \sum_{(\mathbf{m},l):m_1-m_2 \ge -\nu+2\rho_1} W_{\mathbf{m},l}, \quad S^-(\nu) = \sum_{(\mathbf{m},l):m_1-m_2 \le -\nu+2\rho_2} W_{\mathbf{m},l}.$$

Remark

The case $\mathfrak{g} = \mathfrak{su}(d, 1)$, $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})$ can be treated by similar computations; somewhat simpler than the proof of Johnson-Wallach.

Further questions

- Non-compact realization on $L^2(N)$. Different approach using analysis on Heisenberg groups.
- "Cayley" identity for polynomial of CR-Laplacian operators.
- Heisenberg parabolically induced representations for non-Hermitian Lie groups.

$D=G/K$	G	K
$I_{r+b,r}$	$SU(r+b,r))$	$S(U(r+b)\times U(r))$
II_{2r}	$SO^*(4r)$	$U(2r)$
II_{2r+1}	$SO^*(4r+2)$	$U(2r+1)$
III_r	$Sp(r,\mathbb{R})$	$U(r)$
$IV_n, n>4.(r=2)$	$SO(n,2)$	$SO(n)\times SO(2)$
$V(r=2)$	$E_{6(-14)}$	$Spin(10)\times SO(2)$
$VI(r=3)$	$E_{7(-25)}$	$E_6\times SO(2)$

Table 1: Non-compact Hermitian symmetric space $D=G/K$

$D=G/K$	$\mathbb{P}(S)=K/L_0=K_1/L_1$
$I_{r+b,r}$	$I^*_{r+b-1}\times I^*_{r-1}$
II_{2r}	$I^*_{2,2r-2}$
II_{2r+1}	$I^*_{2,2r-1}$
III_r	I^*_{r-1}
$IV_n, n>4$	IV^*_{n-2}
V	II^*_5
VI	V^*

Table 2: The compact Hermitian symmetric space $\mathbb{P}(S)=K/L_0$. D^* is the compact dual of D

Thank you!