Analytic continuation of unitary branching laws for real reductive groups

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Outline

Introduction

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- Proving a branching law includes describing the measure $d\mu$ and its support.

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Casselman embedding Theorem

Every smooth admissible irreducible representation of G occurs in a principal series representation as a quotient.

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For unitary (quotients of) principal series representations π and τ, find symmetry breaking operators

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Prove a Plancherel formula for $f \in \pi^\infty$

$$\|f\|_\pi^2 = \int_{\hat{H}} m(\pi,\tau)\|A_{\pi,\tau}f\|_\tau^2 \, d\mu(\tau).$$

Finite multiplicity pairs of real rank one

[Kobayashi and Oshima, 2013]

If

$$(G, H) = \begin{cases} (\mathrm{O}(1, n+1), \mathrm{O}(1, m+1) \times F), \\ (\mathrm{U}(1, n+1)\, \mathrm{U}(1, m+1) \times F), \\ (\mathrm{Sp}(1, n+1), \mathrm{Sp}(1, m+1) \times F), \\ (F_{4(-20)}, \mathrm{Spin}_0(1, 8) \times F), \end{cases}$$

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- In particular a minimal parabolic of H acts with an open orbit on G/P_G, where P_G is a minimal parabolic of G.

Symmetry breaking operators between principal series representations

Principal series representations

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Definition

For $\xi \in \hat{M}_G$ and $\lambda \in (\mathfrak{a}_G)^*_{\mathbb{C}}$ the principal series representation $\pi_{\xi,\lambda}$ is

$$C^\infty - \mathrm{Ind}_{P_G}^G(\xi \otimes e^\lambda \otimes \mathbf{1}),$$

which is given by left-regular action on the C^∞-sections of the homogeneous bundle

$$G \times_{P_G} (\xi \otimes e^{\lambda+\rho_G} \otimes \mathbf{1}) \to G/P_G.$$

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Let H be a reductive subgroup of G with maximal compact subgroup K_H. Similarly for a minimal parabolic $P_H = M_H \exp(\mathfrak{a}_H) N_H$ of H we define the principal series representations $\tau_{\eta,\nu}$ as the C^∞-sections of the bundle

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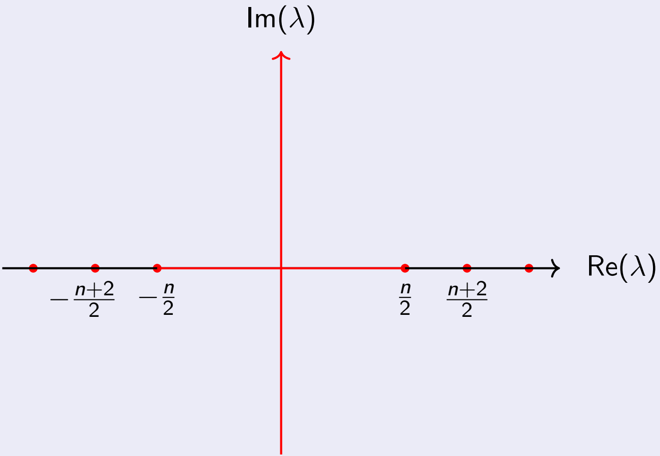
Example

Let $G = O(1, n+1)$. Then $\dim \mathfrak{a}_G = 1$. Let $\lambda \in (\mathfrak{a}_G)^*_{\mathbb{C}}$ and $\xi = \mathbf{1}$.



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- except in the cases $(G, H) = (\mathrm{U}(1, n+1), \mathrm{U}(1,1) \times F), (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1) \times F)$, where additional sporadic operators occur.

Decomposition of unitary representations

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$$\pi_{\xi,\lambda}|_H \to C^\infty(H/H_{x_0}, \mathcal{V}_{\xi,\lambda}),$$

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Idea

Use harmonic analysis on the H-space H/H_{x_0} to decompose unitary closures of $C^\infty(H/H_{x_0}, \mathcal{V}_{\xi,\lambda})$ for each open H-orbit.

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Harmonic analysis on the open orbits

Let $(G, H) = (\mathrm{U}(1, n+1; \mathbb{F}), \mathrm{U}(1, n; \mathbb{F}))$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

- There is one open H-orbit $\mathcal{O}$ and H_{x_0} is a compact subgroup of K_H.

Lemma

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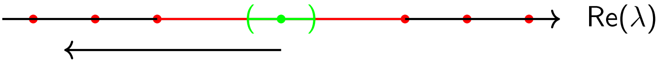
Goal

Decompose $L^2(H/H_{x_0}, \mathcal{V}_{\xi,\lambda})$ and use analytic continuation in λ to expand from $(-R, R)$ towards other unitarizable representations.



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- $H_{x_0} = \mathrm{O}(n) \subseteq K_H = \mathrm{O}(1) \times \mathrm{O}(n)$.
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$$\Phi_\lambda = \Phi_\lambda^+ + \Phi_\lambda^- : \pi_{\mathbf{1},\lambda}|_H \to L^2(H/H_{x_0}) \cong L^2(H/K_H) \oplus L^2(H/K_H, \mathrm{sgn}).$$

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- By classical results on Riemannian symmetric spaces

$$L^2(H/K_H) = \int_{i\mathbb{R}_+}^{\oplus} \hat{\tau}_{\mathbf{1},\nu} \, d\mu(\nu),$$

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- By construction $A_\nu \circ \Phi_\lambda^+ \in \mathrm{Hom}_H(\pi_{\mathbf{1},\lambda}|_H, \tau_{\mathbf{1},\nu})$.

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$$A_\nu \circ \Phi_\lambda^+ = c(\lambda,\nu) A_{\lambda,\nu},$$

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- Recall $\langle f, g\rangle_{\mathbf{1},\lambda} = \langle f, T_{\mathbf{1},\lambda} g\rangle_{L^2(K_G)}$, $T_{\mathbf{1},\lambda} : \pi_{\mathbf{1},\lambda} \to \pi_{\mathbf{1},-\lambda}$.

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- For $\mathrm{Re}(\lambda) \in (-\frac{1}{2}, \frac{1}{2})$, $f \in \pi_{\mathbf{1},\lambda}$ (assuming $\Phi_\lambda^- f = 0$)

$$\begin{aligned}
\|f\|_{\mathbf{1},\lambda}^2 &= \langle f, T_{\mathbf{1},\lambda} f \rangle_{L^2(K_G)} = \langle \Phi_\lambda f, \Phi_{-\lambda} \circ T_{\mathbf{1},\lambda} f \rangle_{L^2(H/H_{x_0})} \\
&= \int_{i\mathbb{R}} \langle A_\nu \circ \Phi_\lambda^+ f, A_\nu \circ \Phi_{-\lambda}^+ \circ T_{\mathbf{1},\lambda} f \rangle_{L^2(K_H)} \frac{d\nu}{c(\nu) c(-\nu)} \\
&= \int_{i\mathbb{R}} \langle A_{\lambda,\nu} f, A_{-\lambda,\nu} \circ T_{\mathbf{1},\lambda} f \rangle_{L^2(K_H)} \frac{c(\lambda,\nu) c(-\lambda,\nu)}{c(\nu) c(-\nu)} d\nu \\
&= t(\lambda) \int_{i\mathbb{R}} \langle A_{\lambda,\nu} f, A_{\lambda,\nu} f \rangle_{L^2(K_H)} \frac{c(\lambda,\nu) c(-\lambda,\nu)}{c(\nu) c(-\nu)} d\nu
\end{aligned}$$

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Theorem

For the complementary series $\lambda \in (-\frac{n}{2}, 0)$

$$\hat{\pi}_{\mathbf{1},\lambda}|_H \simeq \int_{i\mathbb{R}_+}^{\oplus} \hat{\tau}_{\mathbf{1},\nu}\, d\nu \oplus \bigoplus_{k \in [0, \frac{-2\lambda-1}{4}) \cap \mathbb{Z}} \hat{\tau}_{\mathbf{1},\lambda+\frac{1}{2}+2k}$$
$$\oplus \int_{i\mathbb{R}_+}^{\oplus} \hat{\tau}_{\mathrm{sgn},\nu}\, d\nu \oplus \bigoplus_{k \in [0, \frac{-2\lambda-3}{4}) \cap \mathbb{Z}} \hat{\tau}_{\mathrm{sgn},\lambda+\frac{1}{2}+2k}$$

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Recall that the factor $\Gamma((2\lambda - 2\nu + 1)/4)$ of $c(\lambda, \nu)$.

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we can prove Plancherel formulas and unitary branching laws for $\pi_{\xi,\lambda}|_H$ by analytic continuation.

Results

Let $(G, H) = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$. Then $M_G = \mathrm{O}(1) \times \mathrm{O}(n)$. Let $\xi = \alpha \otimes \bigwedge^p(\mathbb{C}^n)$, $\alpha \in \hat{\mathrm{O}}(1)$.

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- [Speh and Zhang, 2016]

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