Multiplicities of stable eigenvalues of compact anti-de Sitter 3-manifolds

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Outline

1. Main result 1
2. Main result 2
3. Idea of proof

Outline

Our setting

$$\Gamma \underset{\text{discrete}}{\subset} G = \mathrm{SO}(2,2) \supset H = \mathrm{SO}(2,1)$$

$$\rightsquigarrow \Gamma \curvearrowright G/H$$

Assume Γ is a discontinuous group for G/H.
Furthermore, for simplicity, suppose Γ is torsion-free.

Our interest

Spectral properties of the differential operator $\Box$ on $\Gamma\backslash G/H$

$$\Box := \frac{1}{8}(\text{the Casimir element for the Killing form of } \mathfrak{g})$$

Discontinuous groups for AdS^3

$$\Gamma \underset{\text{discrete}}{\subset} G = \mathrm{SO}(2,2) \supset H = \mathrm{SO}(2,1)$$

$$\rightsquigarrow \Gamma \curvearrowright G/H$$

Definition (Toshiyuki Kobayashi)

Γ: discontinuous group for $G/H \overset{\text{def}}{\Longleftrightarrow} \Gamma \curvearrowright G/H$: properly discontinuous

discontinuous gp. $\neq$ discrete subgp.

Γ: a discontinuous group for G/H

- $\Gamma\backslash G/H$ is a C^∞-mfd
- the quot. map. $G/H \to \Gamma\backslash G/H$ is a covering.

Anti-de Sitter 3-manifolds

Anti-de Sitter 3-mfd. $\overset{\text{def}}{=}$ Lorentz 3-mfd. of const. sect. curv. $\equiv -1$

Examples

- $\mathrm{AdS}^3 := \mathrm{SO}(2,2)/\mathrm{SO}(2,1)$
  $+$ the metric of sign $(2,1)$ induced by $\frac{1}{8}$(the Killing form of $\mathfrak{so}(2,2)$)
- $\Gamma\backslash\mathrm{AdS}^3$ (Γ: discont. gp. for AdS^3)

Fact (Kulkarni-Raymond + Klingler)

M: compact anti-de Sitter 3-mfd
$\rightsquigarrow \exists$ discont. gp. Γ for AdS^3, $M \cong \Gamma\backslash\mathrm{AdS}^3$ up to finite covering

Hyperbolic Laplacian

Fact

$\Box := \frac{1}{8}$(the Casimir element for the Killing form of $\mathfrak{g}$)

$\rightsquigarrow$ the Laplacian of the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$

(M, g): pseudo-Riemannian mfd.

The Laplacian $\Box := \mathrm{div} \circ \mathrm{grad} \left(= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right) \right)$

M	Riemann	Lorentz
$\Box$	elliptic	hyperbolic

Multiplicity of discrete spectrum

Discrete spectrum (Kassel-Kobayashi)

(M, g): pseudo-Riemannian mfd.

- $L^2_\lambda(M) := \{f \in L^2(M) \mid \Box f = \lambda f$ in the weak sense$\}$
- $\mathrm{Spec}_d(\Box) := \{\lambda \in \mathbb{C} \mid L^2_\lambda(M) \neq 0\}$

Multiplicity of discrete spectrum

$$\mathcal{N}_M(\lambda) := \dim_{\mathbb{C}} L^2_\lambda(M)$$

$\mathcal{N}_M(\lambda)$ for compact M

M	Riemann	Lorentz
$\mathcal{N}_M(\lambda)$	$< \infty$	?

Standard anti-de Sitter manifolds

$$\mathrm{U}(1,1)/\mathrm{U}(1)\times\{1\} \cong \mathrm{SO}(2,2)/\mathrm{SO}(2,1) = \mathrm{AdS}^3$$

Proposition (Kulkarni, Kobayashi)

Assume $\Gamma \subset \mathrm{U}(1,1)(\subset \mathrm{SO}(2,2))$

- discrete in $\mathrm{U}(1,1) \Leftrightarrow \Gamma \curvearrowright \mathrm{AdS}^3$ is properly discontinuously
- cocompact in $\mathrm{U}(1,1) \Leftrightarrow \Gamma \curvearrowright \mathrm{AdS}^3$ is cocompact

$\rightsquigarrow$ Existence of compact anti-de Sitter mfds

Multiplicity for standard anti-de Sitter manifolds

$\lambda_m := 4m(m-1)$ $(m \in \mathbb{N})$ $\cdots$ discrete spectrum of $\square_{\mathrm{AdS}^3}$

Theorem (Kassel-Kobayashi)

Assume $\Gamma \subset \mathrm{U}(1,1)$.

$$\exists m_0(\Gamma) \in \mathbb{R}, \quad \forall m \geq m_0(\Gamma), \quad \mathcal{N}_{\Gamma \backslash \mathrm{AdS}^3}(\lambda_m) = \infty$$

Fact (Kobayashi '98, Klingler '96)

There are cocompact discontinuous groups Γ for AdS^3
s.t. $\Gamma \subset \mathrm{SO}(2,2)$ are Zariski dense

Main result 1

Main Theorem 1 (K.)

Assume Γ is finitely generated.

$$\exists C(\Gamma) \in \mathbb{R}, \ \ \mathcal{N}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - C(\Gamma)$$

$\Gamma \curvearrowright \mathrm{AdS}^3$ is cocompact $\Rightarrow$ Γ is finitely generated

Corollary

M: compact anti-de Sitter mfd

$$\exists C(M) \in \mathbb{R}, \ \ \mathcal{N}_{M}(\lambda_m) \geq \log_3 m - C(M)$$

Outline

Deformation of discontinuous groups

Deform the embedding $\Gamma \overset{\iota}{\hookrightarrow} G \in \mathrm{Hom}(\Gamma, G)$ (compact-open topology)

discontinuous gp. $\neq$ discrete subgp.

Fact (Kobayashi '98, Klingler '96)

Assume $\Gamma \curvearrowright \mathrm{AdS}^3$ is cocompact.

- (Stability for proper discontinuity)
  $\exists$ nbd. W of ι s.t. $\forall \varphi \in W$, $\varphi(\Gamma) \curvearrowright \mathrm{AdS}^3$ is properly discontinuous
- (Failure of local rigidity)
  the orbit $G \cdot \iota$ by conjugation has no interior points.

Stable eigenvalues of the hyperbolic Laplacian

$\mathcal{U}_\Gamma$: the set of nbds. W of $\iota\colon \Gamma \hookrightarrow G$ s.t. $\forall \varphi \in W$,

- $\varphi(\Gamma) \curvearrowright \mathrm{AdS}^3$ is properly discontinuous;
- φ is injective.

Assume $\Gamma \curvearrowright \mathrm{AdS}^3$ is cocompact. Then $\mathcal{U}_\Gamma \neq \emptyset$ (stability).

Theorem (Kassel-Kobayashi, Adv. Math., 2016)

$\exists m_0(\Gamma) \in \mathbb{R}$, $\exists W \in \mathcal{U}_\Gamma$,

$$\bigcap_{\varphi \in W} \mathrm{Spec}_d(\square_{\varphi(\Gamma)\backslash \mathrm{AdS}^3}) \supset \{\lambda_m \mid m \geq m_0(\Gamma)\}$$

λ_m is a stable eigenvalue of $\square_{\Gamma\backslash \mathrm{AdS}^3}$ under any small deformation of Γ.

Multiplicities of stable eigenvalues

Assume $\Gamma \curvearrowright \mathrm{AdS}^3$ is cocompact.

Definition

$$\widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda) := \sup_{W\in\mathcal{U}_\Gamma} \min_{\varphi\in W} \mathcal{N}_{\varphi(\Gamma)\backslash\mathrm{AdS}^3}(\lambda)$$

$\widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda) \neq 0 \Leftrightarrow \lambda$ is a stable eigenvalue.

Main Theorem 2 (K.)

$$\widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - C'(\Gamma)$$

Main Theorem 1: $\exists C(\Gamma) \in \mathbb{R},\ \ \mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - C(\Gamma)$

$C(\Gamma)$ is continuous w.r.t. deformation of Γ

Outline

Γ-average of non-periodic eigenfunctions

$\lambda_m = 4m(m-1)$ $(m \in \mathbb{N})$

Generalized Poincaré series (Kassel-Kobayashi)

Assume $m \geq 2$.

$$\begin{array}{ccc} L^1_{\lambda_m}(\mathrm{AdS}^3)(\neq 0) & \to & L^1_{\lambda_m}(\Gamma\backslash\mathrm{AdS}^3) \\ \psi_m & \mapsto & \psi_m^\Gamma(\Gamma x) := \sum_{\gamma\in\Gamma}\psi_m(\gamma x) \end{array}$$

Fact (Kassel-Kobayashi)

Assume Γ is finitely generated and ψ_m is $\mathrm{SO}(2)\times\mathrm{SO}(2)$-finite.

$$\exists m_0(\Gamma) \in \mathbb{R}, \quad \forall m \geq m_0(\Gamma), \quad \psi_m^\Gamma \in L^2_{\lambda_m}(\Gamma\backslash\mathrm{AdS}^3)$$

Idea of the proof of Main Theorem 1

Main Theorem 1

Assume Γ is finitely generated.

$$\exists C(\Gamma) \in \mathbb{R}, \ \ \mathcal{N}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - C(\Gamma)$$

Find a family of $\psi_{m,j} \in L^1_{\lambda_m}(\mathrm{AdS}^3)$ s.t. $\psi^{\Gamma}_{m,j}$ are linearly independent.

Idea

Examine the "behavior" of Γ-orbits and eigenfuncs at the origin and at ∞.

Linear independence of generalized Poincaré series

Step 1: take a spherical function $\psi_{m,j} \in L^1_{\lambda_m}(\mathrm{AdS}^3)$ $(j = 0, 1, 2, \dots)$ s.t.

$$|\psi_{m,j}(x)| = \begin{cases} O(t^j) & (t \to 0) \\ O(\exp(-2mt)) & (t \to \infty) \end{cases} \text{ w.r.t geodesic parameter } t$$

$\cdots$ it decays more rapidly at infinity than at the origin.

Step 2: prove the main term of the Γ-average $\psi^{\Gamma}_{m,j}(\Gamma x)$ is $\psi_{m,j}(x)$ for x close to the origin by examining the "behavior" of Γ-orbits:

Kassel-Kobayashi proved the non-vanishing of $\psi^{\Gamma}_{m,0}$.

Linear independence of generalized Poincaré series

Step 2: (the main term of $\psi^{\Gamma}_{m,j}(\Gamma x)) = \psi_{m,j}(x)$ for x close to the origin.

Step 3: prove the lin. indep. of $\psi^{\Gamma}_{m,j}$ $(j = 3, 3^2, \ldots, 3^k)$ for $m \geq \exists m_\Gamma(k)$.

This comes down to the following elementary lemma:

Lemma

$\forall a = (a_1, \ldots, a_k) \in \{\pm 1\}^k,\ \exists \theta_a \in \mathbb{R},\ a_i \cos(3^i \theta_a) > 0\ (i = 1, \ldots, k).$

Example: $k = 5$, $a = (1, 1, 1, -1, 1)$

Choose $\theta_a = 2\pi/3^5$.

$3^i\theta_a$	$2\pi/81$	$2\pi/27$	$2\pi/9$	$2\pi/3$	2π
sign of $\cos(3^i\theta_a)$	$+$	$+$	$+$	$-$	$+$

By calculating $m_\Gamma(k)$,

Main Theorem 1: $\exists C(\Gamma) \in \mathbb{R},\ \ \mathcal{N}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - C(\Gamma)$

Idea of the proof of Main Theorem 2

Main Theorem 1 (Γ: finitely generated)

$$\exists C(\Gamma) \in \mathbb{R}, \quad \mathcal{N}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - C(\Gamma)$$

Main Theorem 2 (multiplicities of stable eigenvalues, $\Gamma \curvearrowright \mathrm{AdS}^3$: cocompact)

$$\widetilde{\mathcal{N}}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - C'(\Gamma)$$

For this, it suffices to show:

$C(\Gamma)$ is continuous w.r.t. deformation of Γ

Idea

$C(\Gamma)$ is defined by using the "behavior" of Γ-orbits at the origin and at infinity.
$\rightsquigarrow$ This depends continuously on Γ!