Toward a Holographic Transform for the Quantum Clebsch-Gordan Formula

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Goal

The focus of today's talk will be fusion rules for finite dimensional representations of the Lie group $SL(2,\mathbb{C})$ and the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$.

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We will henceforth denote a finite dimensional, irreducible $\mathfrak{sl}(2,\mathbb{C})$-module of highest weight n and dimension $n+1$ by $V(n)$. Each such module $V(n)$ is unique up to isomorphism.

Theorem (Clebsch-Gordan)

Let $n \geq m$ be non-negative integers. Then there exists an isomorphism of $\mathfrak{sl}(2,\mathbb{C})$-modules:

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m+2) \oplus V(n-m)$$

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Proof.

It suffices to show for $0 \leq p \leq m$, there exists a highest weight vector w_p of weight $n+m-2p$ in $V(n) \otimes V(m)$.

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Likewise, we may search for a collection of holographic operators:

$$\{\phi_p : V(n+m-2p) \to V(n) \otimes V(m) \mid 0 \leq p \leq m\},$$

which is referred to as a holographic transform.

Uniqueness of Symmetry Breaking Operators and Holographic Operators

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Remark

Since the Clebsch-Gordan formula is multiplicity free, symmetry breaking operators and holographic operators for the Clebsch-Gordan formula will be unique up to multiplicative constants.

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Proposition

Let $\mathbb{C}[x]_{\leq n}$ be the vector space of polynomials of degree less than or equal to n. We define a representation of $\mathfrak{sl}(2,\mathbb{C})$ by the actions:

$$Yp(x) = \frac{d}{dx}p(x)$$

$$Xp(x) = \left(nx - x^2\frac{d}{dx}\right)p(x)$$

$$Hp(x) = \left(2x\frac{d}{dx} - n\right)p(x)$$

for any $p(x) \in \mathbb{C}[x]_{\leq n}$. With this action, we have $\mathbb{C}[x]_{\leq n}$ is an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight n.

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Proposition

As explicit $\mathfrak{sl}(2,\mathbb{C})$-modules, we have:

$$\mathbb{C}_{\leq n}[x] \otimes \mathbb{C}_{\leq m}[y] \cong \mathbb{C}[x,y]_{n,m}$$

where $\mathbb{C}[x,y]_{n,m}$ denotes the vector space of polynomials with degree in x less than n and degree in y less than m.

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Theorem (Molchanov, 2015)

The Poisson transform $M_{n,m,p} : \mathbb{C}[x]_{n+m-2p} \to \mathbb{C}[x,y]_{n,m}$ intertwines the actions of $\mathfrak{sl}(2,\mathbb{C})$ on these polynomial spaces and satisfies:

$$M_{n,m,p}(f(x)) = \sum_{s=0}^{m-p} \binom{m-p}{s} \frac{(n-p+s)!}{(n-p)!} (y-x)^{m-s} \left(\frac{d}{dx}\right)^{m-p-s} f(x)$$

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For some polynomial $f(x,y)$, we set:

$$f^{(a,b)} = \frac{\partial^{a+b} f}{\partial x^a \partial y^b}$$

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The Fourier transform $F_{n,m,p} : \mathbb{C}[x,y]_{n,m} \to \mathbb{C}[x]_{n+m-2p}$ intertwines the actions of $\mathfrak{sl}(2,\mathbb{C})$ on these polynomial spaces and satisfies:

$$F_{n,m,p}(f(x,y)) = \frac{(n+m-2p+1)(n-p+1)!}{(n+m-p+1)!} \times \sum_{\alpha=0}^{p} (-1)^{p-\alpha} \binom{n-p+\alpha}{\alpha} \binom{m-\alpha}{p-\alpha} f^{(j-\alpha,\alpha)}(x,x)$$

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Recall that the Poincaré-Birkhoff-Witt theorem allows to describe $U(\mathfrak{sl}(2,\mathbb{C}))$ as the associative algebra generated by the three elements X, Y, H, subject to the relations:

$$[X,Y] = H, \qquad [H,X] = 2X, \qquad [H,Y] = -2Y \tag{1}$$

and that the set $\{X^i Y^j H^k\}_{i,j,k\in\mathbb{N}}$ constitutes a basis for $U(\mathfrak{sl}(2,\mathbb{C}))$.

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The q-deformed algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is obtained by twisting the relations (1) by the quantum parameter q.

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Proposition (q-Binomial Theorem)

If x and y are variables subject to the relation:

$$yx = q^2 xy$$

Then, for $n \ge 0$:

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k$$

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Definition

Fix a nonzero $q \in \mathbb{C}$, not a root of unity. We define $U_q(\mathfrak{sl}(2,\mathbb{C}))$ as the associative algebra generated by the four variables E, F, K, K^{-1} with the relations:

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The algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is generated by q-deformations of the basis relations defining $U(\mathfrak{sl}(2,\mathbb{C}))$.

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- *Like $U(\mathfrak{sl}(2,\mathbb{C}))$, the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is a Hopf algebra. Unlike $U(\mathfrak{sl}(2,\mathbb{C}))$, $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is not cocommutative.*

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ii. *An alternate formulation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ allows the Hopf algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ to be recovered from the quantum group structure by setting $q = 1$.*

Highest Weight Theory for $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules

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Definition

Let $V^{(q)}$ be a $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module and $\lambda \in \mathbb{C}$. A vector $v \neq 0$ in $V^{(q)}$ is of *weight* λ if $Kv = \lambda v$. If $Kv = \lambda v$ and $Ev = 0$, then v is a *highest weight vector* of weight λ.

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Proposition

Any non-zero finite dimensional $U_q(\mathfrak{sl}(2,\mathbb{C}))$*-module has a highest weight vector.*

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1. *$V^{(q)}$ is generated by a highest weight vector v of weight λ.*
2. *The scalar λ is of the form $\lambda = \varepsilon q^n$ where $\varepsilon = \pm 1$ and $n = \dim(V) - 1$.*

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Theorem

Any finite dimensional $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module is completely reducible, i.e. the direct sum of irreducible $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules.

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Proof.

It suffices to show there exists a highest weight vector $w_p^{(q)}$ of weight q^{n+m-2p} for $0 \leq p \leq m$. $\square$

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One thing the classical and quantum Clebsch-Gordan formulas have in common is their proofs, which are both dependent on highest weight vectors.

Remark

In order to develop a method could be applied to the quantum case, we first focused on the classical case (with Molchanov's Poisson transform as a convenient reference point).

Lemma

Let $V(n)$ and $V(m)$ be $U(\mathfrak{sl}(2,\mathbb{C}))$-modules. Let v be a highest weight vector of $V(n)$ and v' be a highest weight vector of $V(m)$. Denote

$$v_j = \frac{1}{j!} Y^j v \qquad \text{and} \qquad v'_h = \frac{1}{h!} Y^h v'$$

for $0 \leq j \leq n$, $0 \leq h \leq m$.

Lemma

Let $V(n)$ and $V(m)$ be $U(\mathfrak{sl}(2,\mathbb{C}))$-modules. Let v be a highest weight vector of $V(n)$ and v' be a highest weight vector of $V(m)$. Denote

$$v_j = \frac{1}{j!} Y^j v \qquad \text{and} \qquad v'_h = \frac{1}{h!} Y^h v'$$

for $0 \leq j \leq n$, $0 \leq h \leq m$. Then:

$$w_p = \sum_{i=0}^{p} (-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!} v_i \otimes v'_{p-i} \tag{2}$$

is a highest weight vector of $V(n) \otimes V(m)$ of weight $n+m-2p$.

Algebraic Approach to Constructing Classical CG Holographic Transform

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- Consider a map $\phi_{n,m,p}$ such that, if v is the highest weight vector of $V(n+m-2p)$:

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- Extend the map $\phi_{n,m,p}$ to the basis $\frac{1}{j!}Y^j v$ using *equivariance*.
- Express $\phi_{n,m,p}$ with an explicit choice of modules ($\mathbb{C}[x]_{\leq n+m-2p} \to \mathbb{C}[x,y]_{n,m}$).
- Algebraically manipulate the map $\phi_{n,m,p}$ to determine polynomial coefficients of the image of some basis vector (a monomial in x).

Definition

Let $n \geq m \geq p \geq 0$. We define:

$$\phi_{n,m,p} : \mathbb{C}[x]_{\leq n+m-2p} \to \mathbb{C}[x,y]_{n,m}$$
$$x^\ell \mapsto \alpha \sum_{t=\omega_1}^{\omega_2} \beta_t x^{n-t} y^{p+\ell-n+t}$$

where:

$$\omega_1 = \max(0, n-p-\ell) \qquad \omega_2 = \min(n+m-p-\ell, n)$$

$$\alpha = \frac{m!\ell!}{(n+m-2p)!(m-p)!}$$

$$\beta_t = \sum_{i=\psi_1(t)}^{\psi_2(t)} \gamma_i \chi_{i,t-i}$$

$$\psi_1(t) = \max(0, t+2p+\ell-n-m) \qquad \psi_2(t) = \min(p,t)$$

$$\gamma_i = \frac{(-1)^i}{i!(p-i)!}$$

$$\chi_{i,j} = \binom{k}{j} \frac{(n-i)!(m-p+i)!}{(n-i-j)!(m-p-k+i+j)!}$$

Theorem (ES)

The map $\phi_{n,m,p}$ intertwines the actions of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}[x]_{\leq n+m-2p}$ and $\mathbb{C}[x,y]_{n,m}$.

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Theorem (ES)

The Poisson transform $M_{n,m,p}$ and the intertwiner $\phi_{n,m,p}$ satisfy:

$$M_{n,m,p} = \frac{(-1)^p(n+m-2p)!(m-p)!p!}{m!(n-p)!}\phi_{n,m,p} \tag{3}$$

Adapting Approach to Quantum Case: Explicit Realizations

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$$\sigma_a(a) = qa, \qquad \sigma_a(b) = b, \qquad \sigma_b(a) = a, \qquad \sigma_b(b) = qb$$

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Moreover, we can define q-analogues to partial derivatives by:

$$\frac{\partial_q(a^m b^n)}{\partial a} = [m]a^{m-1}b^n \qquad \text{and} \qquad \frac{\partial_q(a^m b^n)}{\partial b} = [n]a^m b^{n-1}$$

Adapting Approach to Quantum Case: Explicit Realizations

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Proposition

Let $\mathbb{C}_q[a,b]_{(n)}$ denote the subspace of $\mathbb{C}_q[a,b]$ which contains polynomials of terms which are homogeneous of degree n.
We define a representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ on $\mathbb{C}_q[a,b]_{(n)}$ by the actions:

$$Ep = a\frac{\partial_q p}{\partial b}, \qquad Fp = \frac{\partial_q p}{\partial a}b$$
$$Kp = (\sigma_a \sigma_b^{-1})(p), \qquad K^{-1}p = \sigma_b \sigma_a^{-1}(p)$$

for any $p(a,b) \in \mathbb{C}_q[a,b]_{(n)}$. We thus have that $\mathbb{C}_q[a,b]_{(n)}$ is an irreducible $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module with highest weight q^n.

Adapting Approach to Quantum Case: Explicit Realizations

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We use:

$$\mathbb{C}_q[a, b, c, d]_{(n,m)}$$

to denote the space of polynomials in a, b, c, d which are homogeneous of degree n in a, b and homogeneous of degree m in c, d and subject to the relations:

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Proposition

We have that, as explicit $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules:

$$\mathbb{C}_q[a,b]_{(n)} \otimes \mathbb{C}_q[c,d]_{(m)} \cong \mathbb{C}_q[a,b,c,d]_{(n,m)}$$

Adapting Approach to Quantum Case: Highest Weight Vector

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Let $n \geq m$, let v'_0 be a highest weight vector of weight q^n in $V_n^{(q)}$, and let v''_0 be a highest weight vector of weight q^m in $V_m^{(q)}$. Let us define

$$v'_j = \frac{1}{[j]!} F^j v'_0 \qquad \text{and} \qquad v''_h = \frac{1}{[h]!} F^h v''_0$$

for $0 \leq j \leq n$, $0 \leq h \leq m$.

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for $0 \leq j \leq n$, $0 \leq h \leq m$. Then, for $0 \leq p \leq m$:

$$w_p^{(q)} = \sum_{i=0}^{p} (-1)^i q^{-i(m-2p+i+1)} \frac{[m-p+i]![n-i]!}{[m-p]![n]!} v'_i \otimes v''_{p-i} \tag{4}$$

is a highest weight vector of weight q^{n+m-2p} in $V_n^{(q)} \otimes V_m^{(q)}$.

Algebraic Construction of Quantum Holographic Transform

Definition

Let $n \geq m \geq p \geq 0$. We define:

$$\phi^{(q)}_{n,m,p} : \mathbb{C}_q[a,b]_{(n+m-2p)} \to \mathbb{C}_q[a,b,c,d]_{(n,m)}$$

$$a^\ell b^k \mapsto \mathrm{A} \sum_{t=\omega_1}^{\omega_2} \mathrm{B}_t a^{n-t} b^t c^{m+n-p-\ell-t} d^{-n+p+\ell+t}$$

where: $\omega_1 = \max(0, n-p-\ell) \qquad \omega_2 = \min(n+m-p-\ell, n)$

$$\mathrm{A} = \frac{[\ell]![m]!}{[n+m-2p]![m-p]!}$$

$$\mathrm{B}_t = \sum_{t=\psi_1(t)}^{\psi_2(t)} \Gamma_i \mathrm{X}_{i,t-i}$$

$$\psi_1(t) = \max(0, t+2p+\ell-n-m) \qquad \psi_2(t) = \min(p,t)$$

$$\Gamma_i = q^{-i(m-2p+i+1)} \frac{(-1)^i}{[i]![p-i]!}$$

$$\mathrm{X}_{i,j} = q^{(2i-n+j)(k-j)} \begin{bmatrix} k \\ j \end{bmatrix} \frac{[n-i]![m-p+i]!}{[n-i-j]![m-p+i-k+j]!}$$

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In the quantum case, however, all we have is our algebraically constructed form.

Conjectural Form for Quantum Holographic Transform

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Conjecture

The map $\phi^{(q)}_{n,m,p}$ may be rewritten such that, for any:

$$f(a,b) \in \mathbb{C}_q[a,b]_{(n+m-2p)}$$

we have:

$$\phi^{(q)}_{n,m,p}(f(a,b)) = \sum_{s=0}^{m-p} \Xi c^s (ac - q^{\zeta} bd)^{m-s} b^{-s} \left(\frac{\partial_q^2}{\partial a \partial b} \right)^{m-p-s} \sigma_b^{\Theta}(f(a,b))$$

where Ξ, ζ, and Θ are constants dependent on n, m, p, and s.

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- This form has the correct range.
- This form is K-equivariant.
- This form is consistent with the known form of the map $\phi^{(q)}_{n,m,p}$ in terms of the powers of a, b, c, d.

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- Continue with our algebraic methods to fill in the gaps of our conjectural form.

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- Extend results about representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ to the infinite dimensional case.
- Explore connections to orthogonal polynomials in the quantum case.

Thanks for listening!

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$p = m$ case

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When $p = m$, we have:

$$\phi^{(q)}_{n,m,m}(a^l b^k) = \frac{[\ell]![m]!}{[n-m]!} \sum_{v=0}^{m} \sum_{i=v}^{\psi_2(k+v)} q^{-i(-m+i+1)+(i-n+k+v)(i-v)}(-1)^i$$
$$\times \frac{[k]![n-i]!}{[v+k-i]![i-v]![m-i]![n-k-v]![v]!} a^{\ell+m-v} b^{n-\ell-m+v} c^v d^{m-v}$$

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If we additionally assume $\ell \leq n - 2m$:

$$\tilde{\phi}^{(q)}_{n,m,m}(a^l b^k) = \frac{[l]![m]!}{[n-m]!} \sum_{v=0}^{m} \sum_{i=v}^{m} q^{-i(-m+i+1)+(i-n+k+v)(i-v)}(-1)^i \times \frac{[k]![n-i]!}{[v+k-i]![i-v]![m-i]![n-k-v]![v]!} a^{l+m-v} b^{n-l-m+v} c^v d^{m-v}$$

Example of Calculations

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When $p = m = 2$, the first ($v = 0$) coefficient of $\tilde{\phi}^{(q)}_{n,m,m}(a^l b^k)$ is given by:

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$$\frac{[l]![2]!}{[n-2]!}\sum_{i=0}^{2}(-1)^i q^{-i(i-1)-2+k+(-l-2+i)(i)}\frac{[n-i]![k]!}{[2-i]![k-i]![i]![n-k]!}$$

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$$= \frac{[l]![2]!}{[n-2]!}\left(q^{-2+k}\frac{[n]![k]!}{[2]![k]![n-k]!} - q^{k-l-3}\frac{[n-1]![k]!}{[k-1]![n-k]!} + q^{k-4-2l}\frac{[n-2]![k]!}{[k-2]![2]![n-k]!}\right)$$

$$= \frac{[n-2-k]!}{[n-2]!}\left(q^{-2+k}\frac{[n]!}{[n-k]!} - q^{2k-n-1}\frac{[2][n-1]![k]!}{[k-1]![n-k]!} + q^{3k-2n}\frac{[n-2]![k]!}{[k-2]![n-k]!}\right)$$

$$= q^{-2+k}\frac{[n][n-1]}{[n-k][n-k-1]} - q^{2k-n-1}\frac{[2][n-1][k]}{[n-k][n-k-1]} + q^{3k-2n}\frac{[k][k-1]}{[n-k][n-k-1]}$$

$$= \frac{1}{[n-k][n-k-1]}(q^{-2+k}[n][n-1] - q^{2k-n-1}[2][n-1][k] + q^{3k-2n}[k][k-1])$$

$$= \frac{1}{[n-k][n-k-1]}(q^{-2+k}\frac{(q^n-q^{-n})(q^{n-1}-q^{-n+1})}{(q-q^{-1})^2}$$

$$- q^{2k-n-1}\frac{(q^2-q^{-2})(q^{n-1}-q^{-n+1})(q^k-q^{-k})}{(q-q^{-1})^3} + q^{3k-2n}\frac{(q^k-q^{-k})(q^{k-1}-q^{-k+1})}{(q-q^{-1})^2})$$

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$$
\begin{aligned}
&= \frac{1}{[n-k][n-k-1]}\Big(\frac{(q-q^{-1})(q^{2n-3+k}-q^{k-3}-q^{k-1}+q^{-2n-1+k})}{(q-q^{-1})^3} \\
&- \frac{(q^{2k}-q^{2k-2n+2}-q^{2k-4}+q^{2k-2n-2})(q^{k}-q^{-k})}{(q-q^{-1})^3} \\
&+ \frac{(q^{1}-q^{-1})(q^{5k-2n-1}-q^{3k-2n-1}-q^{3k-2n+1}+q^{k-2n+1})}{(q-q^{-1})^3}\Big) \\
&= \frac{1}{[n-k][n-k-1]}\Big(\frac{q^{2n-2+k}-q^{k-2}-q^{k}+q^{-2n+k}-q^{2n-4+k}+q^{k-4}+q^{k-2}-q^{-2n-2+k}}{(q-q^{-1})^3} \\
&- \frac{q^{3k}-q^{3k-2n+2}-q^{3k-4}+q^{3k-2n-2}-q^{k}+q^{k-2n+2}+q^{k-4}-q^{k-2n-2}}{(q-q^{-1})^3} \\
&+ \frac{q^{5k-2n}-q^{3k-2n}-q^{3k-2n+2}+q^{k-2n+2}-q^{5k-2n-2}+q^{3k-2n-2}+q^{3k-2n}-q^{k-2n}}{(q-q^{-1})^3}\Big)
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&+ \frac{q^{5k-2n}-q^{3k-2n+2}+q^{k-2n+2}-q^{5k-2n-2}+q^{3k-2n-2}-q^{k-2n}}{(q-q^{-1})^3}\Big) \\
&= \frac{1}{[n-k][n-k-1]}\left(\frac{q^{2n-2+k}-q^{2n-4+k}-q^{3k}+q^{3k-4}+q^{5k-2n}-q^{5k-2n-2}}{(q-q^{-1})^3}\right) \\
&= \frac{q^{2n-2+k}-q^{2n-4+k}-q^{3k}+q^{3k-4}+q^{5k-2n}-q^{5k-2n-2}}{(q^{n-k}-q^{-n+k})(q^{n-k-1}-q^{-n+k+1})(q-q^{-1})} \\
&= \frac{q^{2n-2+k}-q^{2n-4+k}-q^{3k}+q^{3k-4}+q^{5k-2n}-q^{5k-2n-2}}{(q^{2n-2k-1}-q^{-1}-q^{1}+q^{-2n+2k+1})(q-q^{-1})} \\
&= \frac{1}{q^{2-3k}}\left(\frac{q^{2n-2+k}-q^{2n-4+k}-q^{3k}+q^{3k-4}+q^{5k-2n}-q^{5k-2n-2}}{q^{2n-2+k}-q^{3k}+q^{5k-2n}-q^{2n+k-4}+q^{3k-4}-q^{-2n+5k-2}}\right)
\end{aligned}
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\begin{aligned}
&= \frac{1}{[n-k][n-k-1]}\Big(\frac{q^{2n-2+k} - q^{k} + q^{-2n+k} - q^{2n-4+k} + q^{k-4} - q^{-2n-2+k}}{(q-q^{-1})^3} \\
&- \frac{q^{3k} - q^{3k-2n+2} - q^{3k-4} + q^{3k-2n-2} - q^{k} + q^{k-2n+2} + q^{k-4} - q^{k-2n-2}}{(q-q^{-1})^3} \\
&+ \frac{q^{5k-2n} - q^{3k-2n+2} + q^{k-2n+2} - q^{5k-2n-2} + q^{3k-2n-2} - q^{k-2n}}{(q-q^{-1})^3}\Big) \\
&= \frac{1}{[n-k][n-k-1]}\left(\frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{(q-q^{-1})^3}\right) \\
&= \frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{(q^{n-k} - q^{-n+k})(q^{n-k-1} - q^{-n+k+1})(q-q^{-1})} \\
&= \frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{(q^{2n-2k-1} - q^{-1} - q^{1} + q^{-2n+2k+1})(q-q^{-1})} \\
&= \frac{1}{q^{2-3k}}\left(\frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{q^{2n-2+k} - q^{3k} + q^{5k-2n} - q^{2n+k-4} + q^{3k-4} - q^{-2n+5k-2}}\right) \\
&= q^{3k-2}
\end{aligned}
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Proposition (ES)

The map:

$$\tilde{\phi}^{(q)}_{n,m,m} : \mathbb{C}_q[a,b]_{(n-m)} \to \mathbb{C}_q[a,b,c,d]_{(n,m)}$$
$$f(a,b) \mapsto (ac - q^{m-n-1}bd)^m \sigma_b^m(f(a,b))$$

intertwines the actions of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ *on* $\mathbb{C}_q[a,b]_{(n-m)}$ *and* $\mathbb{C}_q[a,b,c,d]_{(n,m)}$*. Thus,* $\tilde{\phi}^{(q)}_{n,m,m}$ *is a quantum Poisson transform in the* $p = m$ *case.*

$p = m$ case

By directly computing coefficients in the $p = m$ case we obtained a form for this specific example of a holographic operator.

Proposition (ES)

The map:

$$\tilde{\phi}^{(q)}_{n,m,m} : \mathbb{C}_q[a,b]_{(n-m)} \to \mathbb{C}_q[a,b,c,d]_{(n,m)}$$
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intertwines the actions of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ on $\mathbb{C}_q[a,b]_{(n-m)}$ and $\mathbb{C}_q[a,b,c,d]_{(n,m)}$. Thus, $\tilde{\phi}^{(q)}_{n,m,m}$ is a quantum Poisson transform in the $p = m$ case.

Conjecture

$$\phi^{(q)}_{n,m,p}(f(a,b)) = \sum_{s=0}^{m-p} \Xi c^s (ac - q^{\zeta} bd)^{m-s} b^{-s} \left(\frac{\partial_q^2}{\partial a \partial b} \right)^{m-p-s} \sigma_b^{\Theta}(f(a,b))$$

where Ξ, ζ, and Θ are constants dependent on n, m, p, and s.