Computation of weighted Bergman inner products on bounded symmetric domains for $SU(r,r)$ and restriction to subgroups (arXiv:2105.13976)

Ryosuke Nakahama

Institute of Mathematics for Industry, Kyushu University

AIM Workshop, Branching Problems and Symmetry Breaking
2021/6/10

Outline

Outline

Weighted Bergman inner product

- $D \subset \mathfrak{p}^+ := M(r, \mathbb{C})$: *Bounded symmetric domain*

$$D := \{x \in M(r, \mathbb{C}) \mid I - xx^* \text{ is positive definite.}\}.$$

- $\widetilde{SU}(r, r)$: The universal covering group of

$$SU(r,r) := \left\{ g \in SL(2r, \mathbb{C}) \,\middle|\, g^* \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} g = \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} \right\}.$$

- Let $\lambda \in \mathbb{C}$. $\widetilde{SU}(r, r)$ acts on $\mathcal{O}(D) = \mathcal{O}_\lambda(D)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(x) := \det(cx+d)^{-\lambda} f\left((ax+b)(cx+d)^{-1}\right).$$

- If $\lambda > 2r - 1$, then preserves the *weighted Bergman inner product*

$$\langle f, g \rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)} \det(I - xx^*)^{\lambda - 2r} dx.$$

$\mathcal{H}_\lambda(D) \subset \mathcal{O}(D)$: The corresponding Hilbert subspace.
$(\tau_\lambda, \mathcal{H}_\lambda(D))$ is called a *holomorphic discrete series representation*.

Weighted Bergman inner product

- $D \subset \mathfrak{p}^+ := M(r, \mathbb{C})$: *Bounded symmetric domain*

$$D := \{x \in M(r, \mathbb{C}) \mid I - xx^* \text{ is positive definite.}\}.$$

- $\widetilde{SU}(r, r)$: The universal covering group of

$$SU(r,r) := \left\{ g \in SL(2r, \mathbb{C}) \,\middle|\, g^* \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} g = \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} \right\}.$$

- Let $\lambda \in \mathbb{C}$. $\widetilde{SU}(r, r)$ acts on $\mathcal{O}(D) = \mathcal{O}_\lambda(D)$ by

$$\tau_\lambda \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(x) := \det(cx + d)^{-\lambda} f\left((ax + b)(cx + d)^{-1}\right).$$

- If $\lambda > 2r - 1$, then preserves the *weighted Bergman inner product*

$$\langle f, g \rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)} \det(I - xx^*)^{\lambda - 2r} dx.$$

$\mathcal{H}_\lambda(D) \subset \mathcal{O}(D)$: The corresponding Hilbert subspace.
$(\tau_\lambda, \mathcal{H}_\lambda(D))$ is called a *holomorphic discrete series representation*.

Weighted Bergman inner product

- $D \subset \mathfrak{p}^+ := M(r, \mathbb{C})$: *Bounded symmetric domain*

$$D := \{x \in M(r, \mathbb{C}) \mid I - xx^* \text{ is positive definite.}\}.$$

- $\widetilde{SU}(r, r)$: The universal covering group of

$$SU(r, r) := \left\{ g \in SL(2r, \mathbb{C}) \,\middle|\, g^* \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} g = \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} \right\}.$$

- Let $\lambda \in \mathbb{C}$. $\widetilde{SU}(r, r)$ acts on $\mathcal{O}(D) = \mathcal{O}_\lambda(D)$ by

$$\tau_\lambda \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(x) := \det(cx + d)^{-\lambda} f\left((ax + b)(cx + d)^{-1} \right).$$

- If $\lambda > 2r - 1$, then preserves the *weighted Bergman inner product*

$$\langle f, g \rangle_\lambda = C_\lambda \int_D f(x) \overline{g(x)} \det(I - xx^*)^{\lambda - 2r} dx.$$

$\mathcal{H}_\lambda(D) \subset \mathcal{O}(D)$: The corresponding Hilbert subspace.
$(\tau_\lambda, \mathcal{H}_\lambda(D))$ is called a *holomorphic discrete series representation*.

Weighted Bergman inner product

- $D \subset \mathfrak{p}^+ := M(r, \mathbb{C})$: *Bounded symmetric domain*

$$D := \{x \in M(r, \mathbb{C}) \mid I - xx^* \text{ is positive definite.}\}.$$

- $\widetilde{SU}(r, r)$: The universal covering group of

$$SU(r, r) := \left\{ g \in SL(2r, \mathbb{C}) \;\middle|\; g^* \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} g = \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} \right\}.$$

- Let $\lambda \in \mathbb{C}$. $\widetilde{SU}(r, r)$ acts on $\mathcal{O}(D) = \mathcal{O}_\lambda(D)$ by

$$\tau_\lambda \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(x) := \det(cx + d)^{-\lambda} f\left((ax + b)(cx + d)^{-1} \right).$$

- If $\lambda > 2r - 1$, then preserves the *weighted Bergman inner product*

$$\langle f, g \rangle_\lambda = C_\lambda \int_D f(x) \overline{g(x)} \det(I - xx^*)^{\lambda - 2r} dx.$$

$\mathcal{H}_\lambda(D) \subset \mathcal{O}(D)$: The corresponding Hilbert subspace.
$(\tau_\lambda, \mathcal{H}_\lambda(D))$ is called a *holomorphic discrete series representation*.

Computation of weighted Bergman inner product

- $\langle f,g\rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)}\det(I-xx^*)^{\lambda-2r}dx$: $\widetilde{SU}(r,r)$-invariant.
- For $f(x), g(x) \in \mathcal{P}(\mathfrak{p}^+) = \mathcal{P}(M(r,\mathbb{C})) = \mathcal{O}(D)_{\widetilde{K}}$, we have

$$\langle f,g\rangle_\lambda = \left\langle f(x), g\left(\frac{\partial}{\partial z}\right) e^{\mathrm{tr}(x^t z)}\Big|_{z=0} \right\rangle_{\lambda,x} = \overline{g\left(\frac{\partial}{\partial z}\right)} \left\langle f(x), e^{\mathrm{tr}(x^t z)} \right\rangle_{\lambda,x}\Big|_{z=0}.$$

  Hence suffices to consider $\left\langle f(x), e^{\mathrm{tr}(x^t z)} \right\rangle_{\lambda,x}$ for $f \in \mathcal{P}(\mathfrak{p}^+)$.
- $\mathcal{P}(\mathfrak{p}^+)$ is decomposed under $K = S(U(r)\times U(r))$ as

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V_{\mathbf{m}}^\vee \boxtimes V_{\mathbf{m}},$$

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1,\ldots,m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\}.$$

Theorem (Ørsted (1980), Faraut–Korányi (1990))

Let $\lambda > 2r-1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$. Then for $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $z \in \mathfrak{p}^+$, we have

$$\left\langle f(x), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^r (\lambda-(j-1))_{m_j}} f(z) \quad ((\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)).$$

Computation of weighted Bergman inner product

- $\langle f,g\rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)}\det(I-xx^*)^{\lambda-2r}dx$: $\widetilde{SU}(r,r)$-invariant.
- For $f(x), g(x) \in \mathcal{P}(\mathfrak{p}^+) = \mathcal{P}(M(r,\mathbb{C})) = \mathcal{O}(D)_{\widetilde{K}}$, we have

$$\langle f,g\rangle_\lambda = \left\langle f(x), g\left(\frac{\partial}{\partial z}\right) e^{\mathrm{tr}(x^t z)}\bigg|_{z=0}\right\rangle_{\lambda,x} = \overline{g\left(\frac{\partial}{\partial z}\right)} \left\langle f(x), e^{\mathrm{tr}(x^t z)}\right\rangle_{\lambda,x}\bigg|_{z=0}.$$

  Hence suffices to consider $\left\langle f(x), e^{\mathrm{tr}(x^t z)}\right\rangle_{\lambda,x}$ for $f \in \mathcal{P}(\mathfrak{p}^+)$.
- $\mathcal{P}(\mathfrak{p}^+)$ is decomposed under $K = S(U(r)\times U(r))$ as

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V_{\mathbf{m}}^\vee \boxtimes V_{\mathbf{m}},$$

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1,\ldots,m_r)\in\mathbb{Z}^r \mid m_1\ge\cdots\ge m_r\ge 0\}.$$

Theorem (Ørsted (1980), Faraut–Korányi (1990))

Let $\lambda > 2r-1$, $\mathbf{m}\in\mathbb{Z}^r_{++}$. Then for $f(x)\in\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $z\in\mathfrak{p}^+$, we have

$$\left\langle f(x), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^r(\lambda-(j-1))_{m_j}} f(z) \quad ((\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)).$$

Computation of weighted Bergman inner product

- $\langle f, g\rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)}\det(I - xx^*)^{\lambda-2r}dx$: $\widetilde{SU}(r,r)$-invariant.
- For $f(x), g(x) \in \mathcal{P}(\mathfrak{p}^+) = \mathcal{P}(M(r,\mathbb{C})) = \mathcal{O}(D)_{\widetilde{K}}$, we have

$$\langle f, g\rangle_\lambda = \left\langle f(x), g\left(\frac{\partial}{\partial z}\right) e^{\operatorname{tr}(x^tz)}\Big|_{z=0} \right\rangle_{\lambda,x} = \overline{g\left(\frac{\partial}{\partial z}\right)} \left\langle f(x), e^{\operatorname{tr}(x^tz)} \right\rangle_{\lambda,x}\Big|_{z=0}.$$

Hence suffices to consider $\left\langle f(x), e^{\operatorname{tr}(x^tz)} \right\rangle_{\lambda,x}$ for $f \in \mathcal{P}(\mathfrak{p}^+)$.

- $\mathcal{P}(\mathfrak{p}^+)$ is decomposed under $K = S(U(r) \times U(r))$ as

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V_{\mathbf{m}}^\vee \boxtimes V_{\mathbf{m}},$$

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\}.$$

Theorem (Ørsted (1980), Faraut–Korányi (1990))

Let $\lambda > 2r - 1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$. Then for $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $z \in \mathfrak{p}^+$, we have

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^r (\lambda - (j-1))_{m_j}} f(z) \quad ((\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)).$$

Computation of weighted Bergman inner product

- $\langle f, g\rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)}\det(I - xx^*)^{\lambda-2r}dx$: $\widetilde{SU}(r,r)$-invariant.
- For $f(x), g(x) \in \mathcal{P}(\mathfrak{p}^+) = \mathcal{P}(M(r,\mathbb{C})) = \mathcal{O}(D)_{\widetilde{K}}$, we have

$$\langle f, g\rangle_\lambda = \left\langle f(x), g\left(\frac{\partial}{\partial z}\right) e^{\operatorname{tr}(x^tz)}\Big|_{z=0}\right\rangle_{\lambda,x} = \overline{g\left(\frac{\partial}{\partial z}\right)} \left\langle f(x), e^{\operatorname{tr}(x^tz)}\right\rangle_{\lambda,x}\Big|_{z=0}.$$

Hence suffices to consider $\left\langle f(x), e^{\operatorname{tr}(x^tz)}\right\rangle_{\lambda,x}$ for $f \in \mathcal{P}(\mathfrak{p}^+)$.

- $\mathcal{P}(\mathfrak{p}^+)$ is decomposed under $K = S(U(r)\times U(r))$ as

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V_{\mathbf{m}}^\vee \boxtimes V_{\mathbf{m}},$$

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1,\dots,m_r) \in \mathbb{Z}^r \mid m_1 \ge \cdots \ge m_r \ge 0\}.$$

Theorem (Ørsted (1980), Faraut–Korányi (1990))

Let $\lambda > 2r-1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$. Then for $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $z \in \mathfrak{p}^+$, we have

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)}\right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^r(\lambda-(j-1))_{m_j}} f(z) \quad ((\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)).$$

Computation of weighted Bergman inner product

- $\langle f,g\rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)}\det(I-xx^*)^{\lambda-2r}dx$: $\widetilde{SU}(r,r)$-invariant.
- For $f(x), g(x) \in \mathcal{P}(\mathfrak{p}^+) = \mathcal{P}(M(r,\mathbb{C})) = \mathcal{O}(D)_{\widetilde{K}}$, we have

$$\langle f,g\rangle_\lambda = \left\langle f(x), g\left(\frac{\partial}{\partial z}\right) e^{\operatorname{tr}(x^t z)}\Big|_{z=0} \right\rangle_{\lambda,x} = \overline{g\left(\frac{\partial}{\partial z}\right)} \left\langle f(x), e^{\operatorname{tr}(x^t z)} \right\rangle_{\lambda,x}\Big|_{z=0}.$$

Hence suffices to consider $\left\langle f(x), e^{\operatorname{tr}(x^t z)} \right\rangle_{\lambda,x}$ for $f \in \mathcal{P}(\mathfrak{p}^+)$.

- $\mathcal{P}(\mathfrak{p}^+)$ is decomposed under $K = S(U(r)\times U(r))$ as

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V_{\mathbf{m}}^\vee \boxtimes V_{\mathbf{m}},$$

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1,\ldots,m_r)\in\mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\}.$$

Theorem (Ørsted (1980), Faraut–Korányi (1990))

Let $\lambda > 2r-1$, $\mathbf{m}\in\mathbb{Z}^r_{++}$. Then for $f(x)\in\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $z\in\mathfrak{p}^+$, we have

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^r (\lambda-(j-1))_{m_j}} f(z) \quad ((\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)).$$

Computation of weighted Bergman inner product

- $\langle f,g\rangle_\lambda = C_\lambda \int_D f(x)\overline{g(x)}\det(I-xx^*)^{\lambda-2r}dx$: $\widetilde{SU}(r,r)$-invariant.
- For $f(x), g(x) \in \mathcal{P}(\mathfrak{p}^+) = \mathcal{P}(M(r,\mathbb{C})) = \mathcal{O}(D)_{\widetilde{K}}$, we have

$$\langle f,g\rangle_\lambda = \left\langle f(x), g\left(\frac{\partial}{\partial z}\right)e^{\operatorname{tr}(x^tz)}\bigg|_{z=0}\right\rangle_{\lambda,x} = \overline{g\left(\frac{\partial}{\partial z}\right)}\left\langle f(x), e^{\operatorname{tr}(x^tz)}\right\rangle_{\lambda,x}\bigg|_{z=0}.$$

Hence suffices to consider $\left\langle f(x), e^{\operatorname{tr}(x^tz)}\right\rangle_{\lambda,x}$ for $f\in\mathcal{P}(\mathfrak{p}^+)$.

- $\mathcal{P}(\mathfrak{p}^+)$ is decomposed under $K = S(U(r)\times U(r))$ as

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V_{\mathbf{m}}^\vee \boxtimes V_{\mathbf{m}},$$

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1,\dots,m_r)\in\mathbb{Z}^r \mid m_1\ge\cdots\ge m_r\ge 0\}.$$

Theorem (Ørsted (1980), Faraut–Korányi (1990))

Let $\lambda > 2r-1$, $\mathbf{m}\in\mathbb{Z}^r_{++}$. Then for $f(x)\in\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $z\in\mathfrak{p}^+$, we have

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)}\right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^r(\lambda-(j-1))_{m_j}} f(z) \quad ((\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)).$$

Aim of this talk

- $\mathfrak{p}_s^+ := \mathrm{Sym}(r,\mathbb{C})$, $\mathfrak{p}_a^+ := \mathrm{Alt}(r,\mathbb{C})$: Symmetric and alternating matrices.
- Spaces of polynomials are decomposed under $U(r)$ as

$$\mathcal{P}(\mathfrak{p}_s^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_s^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} V^\vee_{\underbrace{(2m_1,2m_2,\ldots,2m_r)}_{=:2\mathbf{m}}},$$

$$\mathcal{P}(\mathfrak{p}_a^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_a^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} V^\vee_{\underbrace{(m_1,m_1,m_2,m_2,\ldots,m_{\lfloor r/2\rfloor},m_{\lfloor r/2\rfloor}(,0))}_{=:\mathbf{m}^2}}.$$

- Write $x = x_s + x_a \in \mathfrak{p}^+ = \mathfrak{p}_s^+ \oplus \mathfrak{p}_a^+$.
- For $f(x_s) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_s^+)$ or $f(x_a) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_a^+)$, want to compute

$$\left\langle f(x_s), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}, \qquad \left\langle f(x_a), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}.$$

Aim of this talk

- $\mathfrak{p}_{\mathrm{s}}^+ := \mathrm{Sym}(r,\mathbb{C})$, $\mathfrak{p}_{\mathrm{a}}^+ := \mathrm{Alt}(r,\mathbb{C})$: Symmetric and alternating matrices.
- Spaces of polynomials are decomposed under $U(r)$ as

$$\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{s}}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} V^\vee_{\underbrace{(2m_1,2m_2,\ldots,2m_r)}_{=:2\mathbf{m}}},$$

$$\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} V^\vee_{\underbrace{(m_1,m_1,m_2,m_2,\ldots,m_{\lfloor r/2\rfloor},m_{\lfloor r/2\rfloor}(,0))}_{=:\mathbf{m}^2}}.$$

- Write $x = x_{\mathrm{s}} + x_{\mathrm{a}} \in \mathfrak{p}^+ = \mathfrak{p}_{\mathrm{s}}^+ \oplus \mathfrak{p}_{\mathrm{a}}^+$.
- For $f(x_{\mathrm{s}}) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{s}}^+)$ or $f(x_{\mathrm{a}}) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^+)$, want to compute

$$\left\langle f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}, \qquad \left\langle f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}.$$

Aim of this talk

- $\mathfrak{p}_s^+ := \mathrm{Sym}(r,\mathbb{C})$, $\mathfrak{p}_a^+ := \mathrm{Alt}(r,\mathbb{C})$: Symmetric and alternating matrices.
- Spaces of polynomials are decomposed under $U(r)$ as

$$\mathcal{P}(\mathfrak{p}_s^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_s^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} V^\vee_{\underbrace{(2m_1,2m_2,\ldots,2m_r)}_{=:2\mathbf{m}}},$$

$$\mathcal{P}(\mathfrak{p}_a^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_a^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} V^\vee_{\underbrace{(m_1,m_1,m_2,m_2,\ldots,m_{\lfloor r/2\rfloor},m_{\lfloor r/2\rfloor}(,0))}_{=:\mathbf{m}^2}}.$$

- Write $x = x_s + x_a \in \mathfrak{p}^+ = \mathfrak{p}_s^+ \oplus \mathfrak{p}_a^+$.
- For $f(x_s) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_s^+)$ or $f(x_a) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_a^+)$, want to compute

$$\left\langle f(x_s), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}, \qquad \left\langle f(x_a), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}.$$

Aim of this talk

- $\mathfrak{p}_{\mathrm{s}}^+ := \mathrm{Sym}(r, \mathbb{C})$, $\mathfrak{p}_{\mathrm{a}}^+ := \mathrm{Alt}(r, \mathbb{C})$: Symmetric and alternating matrices.
- Spaces of polynomials are decomposed under $U(r)$ as

$$\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{s}}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^r} V^\vee_{\underbrace{(2m_1,2m_2,\dots,2m_r)}_{=:2\mathbf{m}}},$$

$$\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} V^\vee_{\underbrace{(m_1,m_1,m_2,m_2,\dots,m_{\lfloor r/2\rfloor},m_{\lfloor r/2\rfloor}(,0))}_{=:\mathbf{m}^2}}.$$

- Write $x = x_{\mathrm{s}} + x_{\mathrm{a}} \in \mathfrak{p}^+ = \mathfrak{p}_{\mathrm{s}}^+ \oplus \mathfrak{p}_{\mathrm{a}}^+$.
- For $f(x_{\mathrm{s}}) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{s}}^+)$ or $f(x_{\mathrm{a}}) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^+)$, want to compute

$$\left\langle f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}, \qquad \left\langle f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}.$$

Examples

$\mathfrak{p}^+ = M(r,\mathbb{C}) = \mathfrak{p}_\mathrm{s}^+ \oplus \mathfrak{p}_\mathrm{a}^+ = \mathrm{Sym}(r,\mathbb{C}) \oplus \mathrm{Alt}(r,\mathbb{C})$.

- Consider $f(x_\mathrm{a}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}_\mathrm{a}^+)$ or $f(x_\mathrm{s}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_\mathrm{s}^+)$.
- We can show

$$\begin{aligned} \mathcal{P}_{(\overbrace{1,\ldots,1}^{l},0,\ldots,0)}(\mathfrak{p}_\mathrm{a}^+) &\subset \mathcal{P}_{(\overbrace{1,\ldots,1}^{l},0,\ldots,0)}(\mathfrak{p}^+) \\ \simeq V^\vee_{(\underbrace{1,\ldots,1}_{2l},0,\ldots,0)} & \quad \simeq V^\vee_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)} \boxtimes V_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}, \\ \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_\mathrm{s}^+) &\subset \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+) \\ \simeq V^\vee_{(2l,0,\ldots,0)} & \quad \simeq V^\vee_{(l,0,\ldots,0)} \boxtimes V_{(l,0,\ldots,0)}. \end{aligned}$$

- By Faraut–Korányi (1990),

$$\left\langle f(x_\mathrm{a}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^{l}(\lambda-(j-1))} f(z_\mathrm{a}) \quad \left(f(x_\mathrm{a}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}_\mathrm{a}^+)\right),$$

$$\left\langle f(x_\mathrm{s}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_l} f(z_\mathrm{s}) \quad \left(f(x_\mathrm{s}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_\mathrm{s}^+)\right).$$

Examples

$\mathfrak{p}^+ = M(r,\mathbb{C}) = \mathfrak{p}^+_{\mathrm{s}} \oplus \mathfrak{p}^+_{\mathrm{a}} = \mathrm{Sym}(r,\mathbb{C}) \oplus \mathrm{Alt}(r,\mathbb{C})$.

- Consider $f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}})$ or $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}})$.
- We can show

$$\begin{aligned} \mathcal{P}_{(\overbrace{1,\ldots,1}^{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}}) &\subset \mathcal{P}_{(\overbrace{1,\ldots,1}^{l},0,\ldots,0)}(\mathfrak{p}^+) \\ \simeq V^\vee_{(\underbrace{1,\ldots,1}_{2l},0,\ldots,0)} & \qquad \simeq V^\vee_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)} \boxtimes V_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}, \\ \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}}) &\subset \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+) \\ \simeq V^\vee_{(2l,0,\ldots,0)} & \qquad \simeq V^\vee_{(l,0,\ldots,0)} \boxtimes V_{(l,0,\ldots,0)}. \end{aligned}$$

- By Faraut–Korányi (1990),

$$\left\langle f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^{l}(\lambda-(j-1))} f(z_{\mathrm{a}}) \quad \left(f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}})\right),$$

$$\left\langle f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_l} f(z_{\mathrm{s}}) \qquad \left(f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}})\right).$$

Examples

$\mathfrak{p}^+ = M(r,\mathbb{C}) = \mathfrak{p}^+_{\mathrm{s}} \oplus \mathfrak{p}^+_{\mathrm{a}} = \mathrm{Sym}(r,\mathbb{C}) \oplus \mathrm{Alt}(r,\mathbb{C})$.

- Consider $f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}})$ or $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}})$.
- We can show

$$\begin{aligned} \mathcal{P}_{(\overbrace{1,\ldots,1}^{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}}) &\subset \mathcal{P}_{(\overbrace{1,\ldots,1}^{l},0,\ldots,0)}(\mathfrak{p}^+) \\ \simeq V^\vee_{(\underbrace{1,\ldots,1}_{2l},0,\ldots,0)} & \quad \simeq V^\vee_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)} \boxtimes V_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}, \\ \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}}) &\subset \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+) \\ \simeq V^\vee_{(2l,0,\ldots,0)} & \quad \simeq V^\vee_{(l,0,\ldots,0)} \boxtimes V_{(l,0,\ldots,0)}. \end{aligned}$$

- By Faraut–Korányi (1990),

$$\left\langle f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^{l}(\lambda-(j-1))} f(z_{\mathrm{a}}) \quad \left(f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}})\right),$$

$$\left\langle f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_l} f(z_{\mathrm{s}}) \qquad \left(f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}})\right).$$

Goal of this talk

$$\mathfrak{p}^+ = M(r,\mathbb{C}) = \mathfrak{p}^+_{\mathrm{s}} \oplus \mathfrak{p}^+_{\mathrm{a}} = \mathrm{Sym}(r,\mathbb{C}) \oplus \mathrm{Alt}(r,\mathbb{C}).$$

$$\left\langle f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^{l}(\lambda-(j-1))} f(z_{\mathrm{a}}) \quad (f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}})),$$

$$\left\langle f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_l} f(z_{\mathrm{s}}) \quad (f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}})).$$

In this talk, more generally we want to consider the inner products for

$$f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{k+1,\ldots,k+1}_{l},\underbrace{k,\ldots,k}_{s-l})}(\mathfrak{p}^+_{\mathrm{a}}) = \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},\underbrace{0,\ldots,0}_{s-l})}(\mathfrak{p}^+_{\mathrm{a}})\,\mathrm{Pf}(x_{\mathrm{a}})^k \quad (r=2s),$$

$$f(x_{\mathrm{s}}) \in \mathcal{P}_{(k+l,\underbrace{k,\ldots,k}_{r-1})}(\mathfrak{p}^+_{\mathrm{s}}) = \mathcal{P}_{(l,\underbrace{0,\ldots,0}_{r-1})}(\mathfrak{p}^+_{\mathrm{s}})\,\det(x_{\mathrm{s}})^k \quad (r:\ \text{general}).$$

Goal of this talk

$$\mathfrak{p}^+ = M(r,\mathbb{C}) = \mathfrak{p}^+_{\mathrm{s}} \oplus \mathfrak{p}^+_{\mathrm{a}} = \mathrm{Sym}(r,\mathbb{C}) \oplus \mathrm{Alt}(r,\mathbb{C}).$$

$$\left\langle f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{\prod_{j=1}^{l}(\lambda-(j-1))} f(z_{\mathrm{a}}) \quad (f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{a}})),$$

$$\left\langle f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_l} f(z_{\mathrm{s}}) \quad (f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}})).$$

In this talk, more generally we want to consider the inner products for

$$f(x_{\mathrm{a}}) \in \mathcal{P}_{(\underbrace{k+1,\ldots,k+1}_{l},\underbrace{k,\ldots,k}_{s-l})}(\mathfrak{p}^+_{\mathrm{a}}) = \mathcal{P}_{(\underbrace{1,\ldots,1}_{l},\underbrace{0,\ldots,0}_{s-l})}(\mathfrak{p}^+_{\mathrm{a}})\,\mathrm{Pf}(x_{\mathrm{a}})^k \quad (r=2s),$$

$$f(x_{\mathrm{s}}) \in \mathcal{P}_{(k+l,\underbrace{k,\ldots,k}_{r-1})}(\mathfrak{p}^+_{\mathrm{s}}) = \mathcal{P}_{(l,\underbrace{0,\ldots,0}_{r-1})}(\mathfrak{p}^+_{\mathrm{s}})\,\det(x_{\mathrm{s}})^k \quad (r\colon \text{general}).$$

Outline

Main theorem

- $\mathfrak{p}^{\pm} := M(r,\mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{s}} := \mathrm{Sym}(r,\mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{a}} := \mathrm{Alt}(r,\mathbb{C})$. $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are mutually dual.
- $x = x_{\mathrm{s}} \oplus x_{\mathrm{a}} \in \mathfrak{p}^{\pm} = \mathfrak{p}^{\pm}_{\mathrm{s}} \oplus \mathfrak{p}^{\pm}_{\mathrm{a}}$.
- $(\circ, \bullet) \in \{(\mathrm{a},\mathrm{s}), (\mathrm{s},\mathrm{a})\}$.
- For $k \in \mathbb{C}$, $l \in \mathbb{Z}_{\geq 0}$, $\underline{k}_l := (\underbrace{k,\ldots,k}_{l}) \in \mathbb{C}^l$.

Theorem (N)

Let $k \in \mathbb{Z}_{\geq 0}$, and let

$$\begin{array}{llll} r = 2s, & 0 \leq l < s, & f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}}), & \varepsilon = 2 \quad (\bullet = \mathrm{a}), \\ r : \text{general}, & l \in \mathbb{Z}_{\geq 0}, & f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}}), & \varepsilon = 1 \quad (\bullet = \mathrm{s}). \end{array}$$

Then for $\mathrm{Re}\,\lambda > 2r-1$ and for $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \mathfrak{p}^+$, we have

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C^r_\bullet(\lambda,k,l) F^l_\bullet \left(\begin{matrix} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{matrix} ; f; z \right).$$

Main theorem

- $\mathfrak{p}^{\pm} := M(r, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{s}} := \mathrm{Sym}(r, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{a}} := \mathrm{Alt}(r, \mathbb{C})$.
  $\mathfrak{p}^{+}$ and $\mathfrak{p}^{-}$ are mutually dual.
- $x = x_{\mathrm{s}} \oplus x_{\mathrm{a}} \in \mathfrak{p}^{\pm} = \mathfrak{p}^{\pm}_{\mathrm{s}} \oplus \mathfrak{p}^{\pm}_{\mathrm{a}}$.
- $(\circ, \bullet) \in \{(\mathrm{a}, \mathrm{s}), (\mathrm{s}, \mathrm{a})\}$.
- For $k \in \mathbb{C}$, $l \in \mathbb{Z}_{\geq 0}$, $\underline{k}_l := (\underbrace{k, \ldots, k}_{l}) \in \mathbb{C}^l$.

Theorem (N)

Let $k \in \mathbb{Z}_{\geq 0}$*, and let*

$$\begin{array}{lllll} r = 2s, & 0 \leq l < s, & f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^{+}_{\mathrm{a}}), & \varepsilon = 2 & (\bullet = \mathrm{a}), \\ r: \textit{general}, & l \in \mathbb{Z}_{\geq 0}, & f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^{+}_{\mathrm{s}}), & \varepsilon = 1 & (\bullet = \mathrm{s}). \end{array}$$

Then for $\mathrm{Re}\,\lambda > 2r - 1$ *and for* $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \mathfrak{p}^{+}$*, we have*

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda, x} = C^r_\bullet(\lambda, k, l) F^l_\bullet \left(\begin{matrix} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{matrix} ; f; z \right).$$

Main theorem — Constants

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C^r_\bullet(\lambda,k,l) F^l_\bullet\left({-\frac{k}{\varepsilon} \atop -\lambda-\frac{2k}{\varepsilon}+r} ; f; z \right).$$

For $\lambda \in \mathbb{C}$, $\mathbf{n} \in \mathbb{C}^r$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$(\lambda+\mathbf{n})_{\mathbf{m}} := \prod_{j=1}^r (\lambda+n_j-(j-1))_{m_j}, \qquad (\lambda)_{\mathbf{m}} := \prod_{j=1}^r (\lambda-(j-1))_{m_j},$$

where $(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1)$. Then

$$C^{2s}_{\mathrm{a}}(\lambda,k,l) = \frac{\left(\lambda - s + \left(\underline{\lfloor \frac{k}{2} \rfloor + \frac{1}{2}}_l, \underline{\lceil \frac{k}{2} \rceil - \frac{1}{2}}_{s-l}\right)\right)_{(\underline{\lceil k/2 \rceil}_l, \underline{\lfloor k/2 \rfloor}_{s-l})}}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_l, \underline{\lfloor k/2 \rfloor}_{s-l})}},$$

$$C^r_{\mathrm{s}}(\lambda,k,l) = \begin{cases} \dfrac{\left(\lambda + k - \frac{r}{2} + \frac{1}{2} + (l, \underline{0}_{r/2-1})\right)_{\underline{k}_{r/2}}}{(\lambda)_{(2k+l, \underline{2k}_{r/2-1}, \underline{k}_{r/2})}} & (r\colon \text{even}), \\ \dfrac{\left(\lambda - \lfloor \frac{r}{2} \rfloor + \max\{2k, k+l\}\right)_{\min\{k,l\}} \left(\lambda + k - \lceil \frac{r}{2} \rceil + \frac{1}{2}\right)_{\underline{k}_{\lfloor r/2 \rfloor}}}{(\lambda)_{(2k+l, \underline{2k}_{\lfloor r/2 \rfloor -1}, \min\{2k,k+l\}, \underline{k}_{\lfloor r/2 \rfloor})}} & (r\colon \text{odd}). \end{cases}$$

Main theorem — Constants

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = C_\bullet^r(\lambda,k,l) F_\bullet^l \left({-\frac{k}{\varepsilon} \atop -\lambda-\frac{2k}{\varepsilon}+r} ; f; z \right).$$

For $\lambda \in \mathbb{C}$, $\mathbf{n} \in \mathbb{C}^r$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$(\lambda+\mathbf{n})_{\mathbf{m}} := \prod_{j=1}^{r} (\lambda + n_j - (j-1))_{m_j}, \qquad (\lambda)_{\mathbf{m}} := \prod_{j=1}^{r} (\lambda - (j-1))_{m_j},$$

where $(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1)$. Then

$$C_{\mathrm{a}}^{2s}(\lambda,k,l) = \frac{\left(\lambda - s + \left(\underline{\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}}_l, \underline{\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}}_{s-l}\right)\right)_{(\underline{\lceil k/2 \rceil}_l, \underline{\lfloor k/2 \rfloor}_{s-l})}}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_l, \underline{\lfloor k/2 \rfloor}_{s-l})}},$$

$$C_{\mathrm{s}}^{r}(\lambda,k,l) = \begin{cases} \dfrac{\left(\lambda + k - \frac{r}{2} + \frac{1}{2} + (l, \underline{0}_{r/2-1})\right)_{\underline{k}_{r/2}}}{(\lambda)_{(2k+l, \underline{2k}_{r/2-1}, \underline{k}_{r/2})}} & (r: \text{even}), \\[2ex] \dfrac{\left(\lambda - \left\lfloor \frac{r}{2} \right\rfloor + \max\{2k, k+l\}\right)_{\min\{k,l\}} \left(\lambda + k - \left\lceil \frac{r}{2} \right\rceil + \frac{1}{2}\right)_{\underline{k}_{\lfloor r/2 \rfloor}}}{(\lambda)_{(2k+l, \underline{2k}_{\lfloor r/2 \rfloor - 1}, \min\{2k,k+l\}, \underline{k}_{\lfloor r/2 \rfloor})}} & (r: \text{odd}). \end{cases}$$

Main theorem — Constants

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = C^r_\bullet(\lambda,k,l) F^l_\bullet\left({-\frac{k}{\varepsilon} \atop -\lambda-\frac{2k}{\varepsilon}+r} ; f; z \right).$$

For $\lambda \in \mathbb{C}$, $\mathbf{n} \in \mathbb{C}^r$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$(\lambda+\mathbf{n})_{\mathbf{m}} := \prod_{j=1}^r (\lambda + n_j - (j-1))_{m_j}, \qquad (\lambda)_{\mathbf{m}} := \prod_{j=1}^r (\lambda - (j-1))_{m_j},$$

where $(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1)$. Then

$$C^{2s}_{\mathrm{a}}(\lambda,k,l) = \frac{\left(\lambda - s + \left(\underline{\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}}_l, \underline{\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}}_{s-l}\right)\right)_{(\underline{\lceil k/2 \rceil}_l, \underline{\lfloor k/2 \rfloor}_{s-l})}}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_l, \underline{\lfloor k/2 \rfloor}_{s-l})}},$$

$$C^r_{\mathrm{s}}(\lambda,k,l) = \begin{cases} \dfrac{\left(\lambda + k - \frac{r}{2} + \frac{1}{2} + (l, \underline{0}_{r/2-1})\right)_{\underline{k}_{r/2}}}{(\lambda)_{(2k+l, \underline{2k}_{r/2-1}, \underline{k}_{r/2})}} & (r\colon \text{even}), \\ \dfrac{\left(\lambda - \left\lfloor \frac{r}{2} \right\rfloor + \max\{2k, k+l\}\right)_{\min\{k,l\}} \left(\lambda + k - \left\lceil \frac{r}{2} \right\rceil + \frac{1}{2}\right)_{\underline{k}_{\lfloor r/2 \rfloor}}}{(\lambda)_{(2k+l, \underline{2k}_{\lfloor r/2 \rfloor - 1}, \min\{2k, k+l\}, \underline{k}_{\lfloor r/2 \rfloor})}} & (r\colon \text{odd}). \end{cases}$$

Main theorem — $\mathfrak{p}_{\mathrm{a}}^{+} = \mathrm{Alt}(2s, \mathbb{C})$ case

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_\bullet^r(\lambda,k,l) F_\bullet^l \left({-\frac{k}{\varepsilon} \atop -\lambda-\frac{2k}{\varepsilon}+r} ; f; z \right),$$

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^{\pm})$, $\mu,\nu \in \mathbb{C}$, define $F_{\mathrm{a}}^l\left({\nu \atop \mu}; f; z\right) \in C(\Omega)$ $(\exists \Omega \subset \mathfrak{p}^{\pm})$ by

$$F_{\mathrm{a}}^l \left({\nu \atop \mu} ; f; z \right)$$

$$= \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}),$$

where

$$\begin{aligned} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) &\in \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \\ &\subset \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \otimes \mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_l)}(\mathfrak{p}_{\mathrm{a}}^{\mp}), \end{aligned}$$

$$\sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) = e^{\frac{1}{2}\mathrm{tr}(z_{\mathrm{s}} w_{\mathrm{a}} z_{\mathrm{s}} w_{\mathrm{a}})} f({}^t w_{\mathrm{a}}^{-1})$$

$(\mathbf{m}^2 := (m_1, m_1, m_2, m_2, \ldots, m_s, m_s))$.

Main theorem — $\mathfrak{p}_{\mathrm{a}}^{+} = \mathrm{Alt}(2s, \mathbb{C})$ case

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_\bullet^r(\lambda,k,l) F_\bullet^l\left({-\frac{k}{\varepsilon} \atop -\lambda-\frac{2k}{\varepsilon}+r}; f; z \right),$$

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^{\pm})$, $\mu,\nu \in \mathbb{C}$, define $F_{\mathrm{a}}^l\left({\nu \atop \mu}; f; z\right) \in C(\Omega)$ $(\exists \Omega \subset \mathfrak{p}^{\pm})$ by

$$F_{\mathrm{a}}^l\left({\nu \atop \mu}; f; z\right) = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\frac{1}{2}}_{s-l}, \underline{-\frac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\frac{1}{2}}_{s-l}, \underline{-\frac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}),$$

where

$$F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) \in \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \subset \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \otimes \mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_l)}(\mathfrak{p}_{\mathrm{a}}^{\mp}),$$

$$\sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) = e^{\frac{1}{2}\mathrm{tr}(z_{\mathrm{s}} w_{\mathrm{a}} z_{\mathrm{s}} w_{\mathrm{a}})} f({}^t w_{\mathrm{a}}^{-1})$$

$(\mathbf{m}^2 := (m_1, m_1, m_2, m_2, \ldots, m_s, m_s))$.

Main theorem — $\mathfrak{p}_{\mathrm{a}}^{+} = \mathrm{Alt}(2s, \mathbb{C})$ case

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C^r_\bullet(\lambda,k,l) F^l_\bullet \left({-\frac{k}{\varepsilon} \atop -\lambda-\frac{2k}{\varepsilon}+r} ; f; z \right),$$

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^{\pm})$, $\mu,\nu \in \mathbb{C}$, define $F^l_{\mathrm{a}}\left({\nu \atop \mu}; f; z \right) \in C(\Omega)$ $(\exists \Omega \subset \mathfrak{p}^{\pm})$ by

$$F^l_{\mathrm{a}}\left({\nu \atop \mu}; f; z \right)$$

$$= \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l} \right) \right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l} \right) \right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}),$$

where

$$F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) \in \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \subset \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \otimes \mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_l)}(\mathfrak{p}_{\mathrm{a}}^{\mp}),$$

$$\sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) = e^{\frac{1}{2}\mathrm{tr}(z_{\mathrm{s}} w_{\mathrm{a}} z_{\mathrm{s}} w_{\mathrm{a}})} f({}^t w_{\mathrm{a}}^{-1})$$

$(\mathbf{m}^2 := (m_1, m_1, m_2, m_2, \ldots, m_s, m_s))$.

Main theorem — $\mathfrak{p}_{\mathrm{a}}^{+} = \mathrm{Alt}(2s, \mathbb{C})$ case

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_\bullet^r(\lambda,k,l) F_\bullet^l \left({-\frac{k}{\varepsilon} \atop -\lambda-\frac{2k}{\varepsilon}+r} ; f; z \right),$$

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^{\pm})$, $\mu,\nu\in\mathbb{C}$, define $F_{\mathrm{a}}^l\left({\nu \atop \mu}; f; z\right) \in C(\Omega)$ $(\exists\Omega\subset\mathfrak{p}^{\pm})$ by

$$F_{\mathrm{a}}^l\left({\nu \atop \mu}; f; z\right)$$

$$= \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}),$$

where

$$\begin{aligned} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) &\in \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \\ &\subset \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm}) \otimes \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\mp}) \otimes \mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_l)}(\mathfrak{p}_{\mathrm{a}}^{\mp}), \end{aligned}$$

$$\sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, w_{\mathrm{a}}) = e^{\frac{1}{2}\mathrm{tr}(z_{\mathrm{s}} w_{\mathrm{a}} z_{\mathrm{s}} w_{\mathrm{a}})} f({}^t w_{\mathrm{a}}^{-1})$$

$(\mathbf{m}^2 := (m_1, m_1, m_2, m_2, \ldots, m_s, m_s))$.

Main theorem — $\mathfrak{p}_{\mathrm{s}}^+ = \mathrm{Sym}(r,\mathbb{C})$ case

For $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_{\mathrm{s}}^\pm)$, $\mu,\nu\in\mathbb{C}$, define $F_{\mathrm{s}}^l\left(\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}; f; z\right) \in C(\Omega')$ $(\exists\Omega'\subset\mathfrak{p}^\pm)$ by

$$F_{\mathrm{s}}^l\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right) = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, {}^t z_{\mathrm{s}}^{-1})$$

$$\times \begin{cases} \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}}{\left(\mu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}} & (r\colon \text{even}), \\[2ex] \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\nu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}}{\left(\mu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\mu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}} & (r\colon \text{odd}), \end{cases}$$

where for odd r, for $\mathbf{l} = (l_1,\ldots,l_{\lfloor r/2\rfloor}, l_{\lceil r/2\rceil})$ let $\mathbf{l}' := (l_1,\ldots,l_{\lfloor r/2\rfloor})$, and

$$F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) \in \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^\pm) \otimes \mathcal{P}_{(m_1,m_1-l_1,m_2,m_2-l_2,\ldots,m_{\lfloor r/2\rfloor},m_{\lfloor r/2\rfloor}-l_{\lfloor r/2\rfloor},(-l_{\lceil r/2\rceil}))}(\mathfrak{p}_{\mathrm{s}}^\mp)$$
$$\subset \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^\pm) \otimes \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^\mp) \otimes \mathcal{P}_{(\underline{0}_{r-1},-l)}(\mathfrak{p}_{\mathrm{s}}^\mp),$$

$$\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) = e^{\frac{1}{2}\operatorname{tr}(z_{\mathrm{a}} w_{\mathrm{s}} z_{\mathrm{a}} w_{\mathrm{s}})} f({}^t w_{\mathrm{s}}^{-1}).$$

Main theorem — $\mathfrak{p}_{\mathrm{s}}^{+} = \mathrm{Sym}(r, \mathbb{C})$ case

For $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_{\mathrm{s}}^{\pm})$, $\mu, \nu \in \mathbb{C}$, define $F_{\mathrm{s}}^{l}\left(\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}; f; z\right) \in C(\Omega')$ $(\exists \Omega' \subset \mathfrak{p}^{\pm})$ by

$$F_{\mathrm{s}}^{l}\begin{pmatrix}\nu \\ \mu\end{pmatrix}; f; z\Bigr) = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil} \\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, {}^{t}z_{\mathrm{s}}^{-1})$$

$$\times \begin{cases} \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1}, l)}}{\left(\mu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1}, l)}} & (r\colon \text{even}), \\[2ex] \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\nu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}}{\left(\mu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\mu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}} & (r\colon \text{odd}), \end{cases}$$

where for odd r, for $\mathbf{l} = (l_1, \ldots, l_{\lfloor r/2\rfloor}, l_{\lceil r/2\rceil})$ let $\mathbf{l}' := (l_1, \ldots, l_{\lfloor r/2\rfloor})$, and

$$F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) \in \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\pm}) \otimes \mathcal{P}_{(m_1, m_1-l_1, m_2, m_2-l_2, \ldots, m_{\lfloor r/2\rfloor}, m_{\lfloor r/2\rfloor}-l_{\lfloor r/2\rfloor}, (-l_{\lceil r/2\rceil}))}(\mathfrak{p}_{\mathrm{s}}^{\mp})$$
$$\subset \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\pm}) \otimes \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\mp}) \otimes \mathcal{P}_{(\underline{0}_{r-1}, -l)}(\mathfrak{p}_{\mathrm{s}}^{\mp}),$$

$$\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil} \\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) = e^{\frac{1}{2}\operatorname{tr}(z_{\mathrm{a}} w_{\mathrm{s}} z_{\mathrm{a}} w_{\mathrm{s}})} f({}^{t}w_{\mathrm{s}}^{-1}).$$

Main theorem — $\mathfrak{p}_{\mathrm{s}}^{+} = \mathrm{Sym}(r, \mathbb{C})$ case

For $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_{\mathrm{s}}^{\pm})$, $\mu, \nu \in \mathbb{C}$, define $F_{\mathrm{s}}^{l}\left(\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}; f; z\right) \in C(\Omega')$ $(\exists \Omega' \subset \mathfrak{p}^{\pm})$ by

$$F_{\mathrm{s}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right) = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, {}^{t}z_{\mathrm{s}}^{-1})$$

$$\times \begin{cases} \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1}, l)}}{\left(\mu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1}, l)}} & (r\colon \text{even}), \\[2ex] \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\nu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}}{\left(\mu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\mu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}} & (r\colon \text{odd}), \end{cases}$$

where for odd r, for $\mathbf{l} = (l_1, \ldots, l_{\lfloor r/2\rfloor}, l_{\lceil r/2\rceil})$ let $\mathbf{l}' := (l_1, \ldots, l_{\lfloor r/2\rfloor})$, and

$$F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) \in \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\pm}) \otimes \mathcal{P}_{(m_1, m_1-l_1, m_2, m_2-l_2, \ldots, m_{\lfloor r/2\rfloor}, m_{\lfloor r/2\rfloor}-l_{\lfloor r/2\rfloor}, (-l_{\lceil r/2\rceil}))}(\mathfrak{p}_{\mathrm{s}}^{\mp})$$
$$\subset \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\pm}) \otimes \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\mp}) \otimes \mathcal{P}_{(\underline{0}_{r-1}, -l)}(\mathfrak{p}_{\mathrm{s}}^{\mp}),$$

$$\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) = e^{\frac{1}{2}\mathrm{tr}(z_{\mathrm{a}} w_{\mathrm{s}} z_{\mathrm{a}} w_{\mathrm{s}})} f({}^{t}w_{\mathrm{s}}^{-1}).$$

Main theorem — $\mathfrak{p}_{\mathrm{s}}^{+} = \mathrm{Sym}(r, \mathbb{C})$ case

For $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_{\mathrm{s}}^{\pm})$, $\mu, \nu \in \mathbb{C}$, define $F_{\mathrm{s}}^{l}\left(\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}; f; z\right) \in C(\Omega')$ $(\exists \Omega' \subset \mathfrak{p}^{\pm})$ by

$$F_{\mathrm{s}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right) = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, {}^{t}z_{\mathrm{s}}^{-1})$$

$$\times \begin{cases} \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1}, l)}}{\left(\mu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1}, l)}} & (r\colon \text{even}), \\[2ex] \dfrac{(\nu)_{\mathbf{m}} \left(\nu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\nu - \frac{1}{2}(r-1) - l\right)_{l - l_{\lceil r/2\rceil}}}{\left(\mu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'} \left(\mu - \frac{1}{2}(r-1) - l\right)_{l - l_{\lceil r/2\rceil}}} & (r\colon \text{odd}), \end{cases}$$

where for odd r, for $\mathbf{l} = (l_1, \ldots, l_{\lfloor r/2\rfloor}, l_{\lceil r/2\rceil})$ let $\mathbf{l}' := (l_1, \ldots, l_{\lfloor r/2\rfloor})$, and

$$\begin{aligned} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) &\in \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\pm}) \otimes \mathcal{P}_{(m_1, m_1 - l_1, m_2, m_2 - l_2, \ldots, m_{\lfloor r/2\rfloor}, m_{\lfloor r/2\rfloor} - l_{\lfloor r/2\rfloor}, (-l_{\lceil r/2\rceil}))}(\mathfrak{p}_{\mathrm{s}}^{\mp}) \\ &\subset \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\pm}) \otimes \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\mp}) \otimes \mathcal{P}_{(\underline{0}_{r-1}, -l)}(\mathfrak{p}_{\mathrm{s}}^{\mp}), \end{aligned}$$

$$\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{s}}[f](z_{\mathrm{a}}, w_{\mathrm{s}}) = e^{\frac{1}{2}\operatorname{tr}(z_{\mathrm{a}} w_{\mathrm{s}} z_{\mathrm{a}} w_{\mathrm{s}})} f({}^{t}w_{\mathrm{s}}^{-1}).$$

Poles of inner product — $\mathfrak{p}_{\mathrm{a}}^{+} = \mathrm{Alt}(2s, \mathbb{C})$ case

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{a}}^{2s}(\lambda,k,l) F_{\mathrm{a}}^{l}\left({-\frac{k}{2} \atop -\lambda-k+2s} ; f; z \right),$$

$$C_{\mathrm{a}}^{2s}(\lambda,k,l) = \frac{\left(\lambda - s + \left(\underline{\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}}_{l}, \underline{\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}}_{s-l}\right)\right)_{(\underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})}}{(\lambda)_{(\underline{k+1}_{l}, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})}}.$$

Corollary (N)

When $r = 2s$, for $k \in \mathbb{Z}_{\geq 0}$, $l \in \{0, 1, \ldots, s-1\}$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^{+})$,

$$(\lambda)_{(\underline{k+1}_{l}, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})} \left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$.

Poles of inner product — $\mathfrak{p}_{\mathrm{a}}^{+} = \mathrm{Alt}(2s, \mathbb{C})$ case

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^{k} f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^{*})} \right\rangle_{\lambda,x} = C_{\mathrm{a}}^{2s}(\lambda, k, l) F_{\mathrm{a}}^{l}\left({}^{-\frac{k}{2}}_{-\lambda-k+2s} ; f; z \right),$$

$$C_{\mathrm{a}}^{2s}(\lambda, k, l) = \frac{\left(\lambda - s + \left(\underline{\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}}_{l}, \underline{\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}}_{s-l}\right)\right)_{(\underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})}}{(\lambda)_{(\underline{k+1}_{l}, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})}}.$$

Corollary (N)

When $r = 2s$, for $k \in \mathbb{Z}_{\geq 0}$, $l \in \{0, 1, \ldots, s-1\}$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_{l}}(\mathfrak{p}_{\mathrm{a}}^{+})$,

$$(\lambda)_{(\underline{k+1}_{l}, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})} \left\langle \mathrm{Pf}(x_{\mathrm{a}})^{k} f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^{*})} \right\rangle_{\lambda,x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$.

Poles of inner product — $\mathfrak{p}_{\mathrm{a}}^{+} = \mathrm{Alt}(2s, \mathbb{C})$ case

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{a}}^{2s}(\lambda,k,l) F_{\mathrm{a}}^{l}\left({-\frac{k}{2} \atop -\lambda-k+2s}; f; z \right),$$

$$C_{\mathrm{a}}^{2s}(\lambda,k,l) = \frac{\left(\lambda - s + \left(\underline{\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}}_{l}, \underline{\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}}_{s-l}\right)\right)_{(\underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})}}{(\lambda)_{(\underline{k+1}_{l}, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})}}.$$

Corollary (N)

When $r = 2s$, for $k \in \mathbb{Z}_{\geq 0}$, $l \in \{0, 1, \ldots, s-1\}$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_{l}}(\mathfrak{p}_{\mathrm{a}}^{+})$,

$$(\lambda)_{(\underline{k+1}_{l}, \underline{k}_{s-l}, \underline{\lceil k/2 \rceil}_{l}, \underline{\lfloor k/2 \rfloor}_{s-l})} \left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$.

Poles of inner product — $\mathfrak{p}_{\mathrm{s}}^{+} = \mathrm{Sym}(r, \mathbb{C})$ case

$$\left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{s}}^r(\lambda,k,l) F_{\mathrm{s}}^l \left({-k \atop -\lambda-2k+r}; f; z \right),$$

$$C_{\mathrm{s}}^r(\lambda,k,l) = \begin{cases} \dfrac{\left(\lambda+k-\frac{r}{2}+\frac{1}{2}+(l,\underline{0}_{r/2-1})\right)_{\underline{k}_{r/2}}}{(\lambda)_{(2k+l,\underline{2k}_{r/2-1},\underline{k}_{r/2})}} & (r\colon \text{even}), \\ \dfrac{\left(\lambda-\left\lfloor\frac{r}{2}\right\rfloor+\max\{2k,k+l\}\right)_{\min\{k,l\}} \left(\lambda+k-\left\lceil\frac{r}{2}\right\rceil+\frac{1}{2}\right)_{\underline{k}_{\lfloor r/2\rfloor}}}{(\lambda)_{(2k+l,\underline{2k}_{\lfloor r/2\rfloor-1},\min\{2k,k+l\},\underline{k}_{\lfloor r/2\rfloor})}} & (r\colon \text{odd}). \end{cases}$$

Corollary (N)

For $k, l \in \mathbb{Z}_{\geq 0}$, $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\dots,0)}(\mathfrak{p}_{\mathrm{s}}^{+})$,

$$(\lambda)_{(2k+l,\underline{2k}_{r/2-1},\underline{k}_{r/2})} \left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \quad (\text{if } r\colon \text{even}),$$

$$(\lambda)_{(2k+l,\underline{2k}_{\lfloor r/2\rfloor-1},\min\{k+l,2k\},\underline{k}_{\lfloor r/2\rfloor})} \left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \quad (\text{if } r\colon \text{odd})$$

are holomorphically continued for all $\lambda \in \mathbb{C}$.

Poles of inner product — $\mathfrak{p}_{\mathrm{s}}^{+} = \mathrm{Sym}(r, \mathbb{C})$ case

$$\left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{s}}^r(\lambda,k,l) F_{\mathrm{s}}^l \left({-k \atop -\lambda-2k+r}; f; z \right),$$

$$C_{\mathrm{s}}^r(\lambda,k,l) = \begin{cases} \dfrac{\left(\lambda+k-\frac{r}{2}+\frac{1}{2}+(l,\underline{0}_{r/2-1})\right)_{\underline{k}_{r/2}}}{(\lambda)_{(2k+l,\underline{2k}_{r/2-1},\underline{k}_{r/2})}} & (r\colon \text{even}), \\ \dfrac{\left(\lambda-\left\lfloor\frac{r}{2}\right\rfloor+\max\{2k,k+l\}\right)_{\min\{k,l\}}\left(\lambda+k-\left\lceil\frac{r}{2}\right\rceil+\frac{1}{2}\right)_{\underline{k}_{\lfloor r/2\rfloor}}}{(\lambda)_{(2k+l,\underline{2k}_{\lfloor r/2\rfloor-1},\min\{2k,k+l\},\underline{k}_{\lfloor r/2\rfloor})}} & (r\colon \text{odd}). \end{cases}$$

Corollary (N)

For $k, l \in \mathbb{Z}_{\geq 0}$, $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\dots,0)}(\mathfrak{p}_{\mathrm{s}}^{+})$,

$$(\lambda)_{(2k+l,\underline{2k}_{r/2-1},\underline{k}_{r/2})} \left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \quad (\textit{if } r\colon \textit{even}),$$

$$(\lambda)_{(2k+l,\underline{2k}_{\lfloor r/2\rfloor-1},\min\{k+l,2k\},\underline{k}_{\lfloor r/2\rfloor})} \left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \quad (\textit{if } r\colon \textit{odd})$$

are holomorphically continued for all $\lambda \in \mathbb{C}$.

Poles of inner product — $\mathfrak{p}_{\mathrm{s}}^{+} = \mathrm{Sym}(r, \mathbb{C})$ case

$$\left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{s}}^r(\lambda,k,l) F_{\mathrm{s}}^l \left({-k \atop -\lambda-2k+r}; f; z \right),$$

$$C_{\mathrm{s}}^r(\lambda,k,l) = \begin{cases} \dfrac{\left(\lambda+k-\frac{r}{2}+\frac{1}{2}+(l,\underline{0}_{r/2-1})\right)_{\underline{k}_{r/2}}}{(\lambda)_{(2k+l,\underline{2k}_{r/2-1},\underline{k}_{r/2})}} & (r\colon \text{even}), \\ \dfrac{\left(\lambda-\left\lfloor\frac{r}{2}\right\rfloor+\max\{2k,k+l\}\right)_{\min\{k,l\}}\left(\lambda+k-\left\lceil\frac{r}{2}\right\rceil+\frac{1}{2}\right)_{\underline{k}_{\lfloor r/2\rfloor}}}{(\lambda)_{(2k+l,\underline{2k}_{\lfloor r/2\rfloor-1},\min\{2k,k+l\},\underline{k}_{\lfloor r/2\rfloor})}} & (r\colon \text{odd}). \end{cases}$$

Corollary (N)

For $k,l \in \mathbb{Z}_{\geq 0}$, $f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_{\mathrm{s}}^{+})$,

$$(\lambda)_{(2k+l,\underline{2k}_{r/2-1},\underline{k}_{r/2})} \left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \quad (\text{if } r\colon \text{even}),$$

$$(\lambda)_{(2k+l,\underline{2k}_{\lfloor r/2\rfloor-1},\min\{k+l,2k\},\underline{k}_{\lfloor r/2\rfloor})} \left\langle \det(x_{\mathrm{s}})^k f(x_{\mathrm{s}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \quad (\text{if } r\colon \text{odd})$$

are holomorphically continued for all $\lambda \in \mathbb{C}$.

When $l = 0$, $f(x_\bullet) = 1$

When $l = 0$, $f(x_\bullet) = 1$,

$$F^l_{\mathrm{a}}\begin{pmatrix}\nu\\ \mu\end{pmatrix}; f; z\Big) = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s\\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}}\left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})}^{\substack{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^{\pm}_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}}(\mathfrak{p}^{\mp}_{\mathrm{a}})\\ \otimes\mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_l)}(\mathfrak{p}^{\mp}_{\mathrm{a}})\\ \Cup}},$$

$$F^l_{\mathrm{s}}\begin{pmatrix}\nu\\ \mu\end{pmatrix}; f; z\Big) = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}^{\lfloor r/2\rfloor}_{++}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} \underbrace{F^{\mathrm{s}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{a}}, {}^t z_{\mathrm{s}}^{-1})}_{\substack{\Cap\\ \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}^{\pm}_{\mathrm{a}})\otimes\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^{\mp}_{\mathrm{s}})\otimes\mathcal{P}_{(\underline{0}_{r-1},-l)}(\mathfrak{p}^{\mp}_{\mathrm{s}})}}$$

$$\times \begin{cases} \dfrac{(\nu)_{\mathbf{m}}\left(\nu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}}{\left(\mu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}} & (r\colon \text{even}),\\[2ex] \dfrac{(\nu)_{\mathbf{m}}\left(\nu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\nu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}}{\left(\mu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\mu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}} & (r\colon \text{odd}). \end{cases}$$

When $l = 0$, $f(x_\bullet) = 1$

When $l = 0$, $f(x_\bullet) = 1$,

$$F^l_{\mathrm{a}}\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right) = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s\\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}}\left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})}^{\substack{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^\pm_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}}(\mathfrak{p}^\mp_{\mathrm{a}})\\ \otimes\mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_l)}(\mathfrak{p}^\mp_{\mathrm{a}})\\ \Cup}},$$

$$F^l_{\mathrm{s}}\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right) = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}^{\lfloor r/2\rfloor}_{++}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} \underbrace{F^{\mathrm{s}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{a}}, {}^t z_{\mathrm{s}}^{-1})}_{\substack{\Cap\\ \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}^\pm_{\mathrm{a}})\otimes\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^\mp_{\mathrm{s}})\otimes\mathcal{P}_{(\underline{0}_{r-1},-l)}(\mathfrak{p}^\mp_{\mathrm{s}})}}$$

$$\times\begin{cases} \dfrac{(\nu)_{\mathbf{m}}\left(\nu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}}{\left(\mu - \frac{1}{2} - (\underline{0}_{r/2-1}, l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}} & (r\colon \text{even}),\\[2ex] \dfrac{(\nu)_{\mathbf{m}}\left(\nu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\nu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}}{\left(\mu - \frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\mu - \frac{1}{2}(r-1) - l\right)_{l-l_{\lceil r/2\rceil}}} & (r\colon \text{odd}). \end{cases}$$

When $l = 0$, $f(x_\bullet) = 1$

When $l = 0$, $f(x_\bullet) = 1$,

$$F^{0}_{\mathrm{a}}\binom{\nu}{\mu};1;z = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^{s}_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}}\left(\nu+\left(\underline{\tfrac{1}{2}}_{s-l},\underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l})}}{\left(\mu+\left(\underline{\tfrac{1}{2}}_{s-l},\underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l})}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\underline{0}_{s}}[1](z_{\mathrm{s}},{}^{t}z_{\mathrm{a}}^{-1})}^{\substack{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^{\pm}_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}}(\mathfrak{p}^{\mp}_{\mathrm{a}})\\ \otimes\mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_{l})}(\mathfrak{p}^{\mp}_{\mathrm{a}})\\ \Cup}},$$

$$F^{0}_{\mathrm{s}}\binom{\nu}{\mu};1;z = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}^{\lfloor r/2\rfloor}_{++}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} \underbrace{F^{\mathrm{s}}_{\mathbf{m},\underline{0}_{\lceil r/2\rceil}}[1](z_{\mathrm{a}},{}^{t}z_{\mathrm{s}}^{-1})}_{\substack{\Cap\\ \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}^{\pm}_{\mathrm{a}})\otimes\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^{\mp}_{\mathrm{s}})\otimes\mathcal{P}_{(\underline{0}_{r-1},-l)}(\mathfrak{p}^{\mp}_{\mathrm{s}})}}$$

$$\times\begin{cases} \dfrac{(\nu)_{\mathbf{m}}\left(\nu-\frac{1}{2}-(\underline{0}_{r/2-1},l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}}{\left(\mu-\frac{1}{2}-(\underline{0}_{r/2-1},l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}} & (r\colon \text{even}),\\[2ex] \dfrac{(\nu)_{\mathbf{m}}\left(\nu-\frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\nu-\frac{1}{2}(r-1)-l\right)_{l-l_{\lceil r/2\rceil}}}{\left(\mu-\frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\mu-\frac{1}{2}(r-1)-l\right)_{l-l_{\lceil r/2\rceil}}} & (r\colon \text{odd}). \end{cases}$$

When $l = 0$, $f(x_\bullet) = 1$

When $l = 0$, $f(x_\bullet) = 1$,

$$F_{\mathrm{a}}^{0}\left(\begin{matrix}\nu\\ \mu\end{matrix}; 1; z\right) = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{s}} \sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}}\left(\nu+\left(\underline{\tfrac{1}{2}}_{s-l},\underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l})}}{\left(\mu+\left(\underline{\tfrac{1}{2}}_{s-l},\underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l})}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\underline{0}_{s}}[1](z_{\mathrm{s}},{}^{t}z_{\mathrm{a}}^{-1})}^{\substack{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\pm})\otimes\mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\mp})\\ \otimes\mathcal{P}_{(\underline{0}_{s-l},\underline{-1}_{l})}(\mathfrak{p}_{\mathrm{a}}^{\mp})\\ \cup}},$$

$$F_{\mathrm{s}}^{0}\left(\begin{matrix}\nu\\ \mu\end{matrix}; 1; z\right) = \det(z_{\mathrm{s}})^{-\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\lceil r/2\rceil}\\ |\mathbf{l}|=l}} \underbrace{F^{\mathrm{s}}_{\mathbf{m},\underline{0}_{\lceil r/2\rceil}}[1](z_{\mathrm{a}},{}^{t}z_{\mathrm{s}}^{-1})}_{\substack{\cap\\ \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{\pm})\otimes\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^{\mp})\otimes\mathcal{P}_{(\underline{0}_{r-1},-l)}(\mathfrak{p}_{\mathrm{s}}^{\mp})}}$$

$$\times\begin{cases} \dfrac{(\nu)_{\mathbf{m}}\left(\nu-\frac{1}{2}-(\underline{0}_{r/2-1},l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}}{\left(\mu-\frac{1}{2}-(\underline{0}_{r/2-1},l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{r/2-1},l)}} & (r\colon \text{even}),\\[2ex] \dfrac{(\nu)_{\mathbf{m}}\left(\nu-\frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\nu-\frac{1}{2}(r-1)-l\right)_{l-l_{\lceil r/2\rceil}}}{\left(\mu-\frac{1}{2}\right)_{\mathbf{m}-\mathbf{l}'}\left(\mu-\frac{1}{2}(r-1)-l\right)_{l-l_{\lceil r/2\rceil}}} & (r\colon \text{odd}).\end{cases}$$

When $l = 0$, $f(x_\bullet) = 1$

$$ {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}; z_\circ, w_\bullet\right) := \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} F^\bullet_{\mathbf{m},\underline{0}_{\lceil r/2\rceil}}[1](z_\circ, w_\bullet). $$

Corollary (N)

1. *Let $r = 2s$, $k \in \mathbb{Z}_{\geq 0}$. Then for* $\operatorname{Re}\lambda > 2r-1$, $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \mathfrak{p}^+$, *we have*

$$ \left\langle \operatorname{Pf}(x_{\mathrm{a}})^k, e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{\left(\lambda + \left\lceil \frac{k}{2}\right\rceil - s - \frac{1}{2}\right)_{\underline{\lfloor k/2\rfloor}_s}}{(\lambda)_{\underline{k}_s}(\lambda - s)_{\underline{\lfloor k/2\rfloor}_s}} \operatorname{Pf}(z_{\mathrm{a}})^k {}_2F_1\left(\begin{matrix}-\frac{k}{2}, -\frac{k-1}{2}\\ -\lambda - k + 2s + \frac{1}{2}\end{matrix}; z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}\right). $$

2. *Let r be general, $k \in \mathbb{Z}_{\geq 0}$. Then for* $\operatorname{Re}\lambda > 2r-1$, $z \in \mathfrak{p}^+$, *we have*

$$ \left\langle \det(x_{\mathrm{s}})^k, e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{\left(\lambda + k - \left\lceil \frac{r}{2}\right\rceil + \frac{1}{2}\right)_{\underline{k}_{\lfloor r/2\rfloor}}}{(\lambda)_{\underline{2k}_{\lfloor r/2\rfloor}}\left(\lambda - \left\lfloor \frac{r}{2}\right\rfloor\right)_{\underline{k}_{\lceil r/2\rceil}}} \det(z_{\mathrm{s}})^k {}_2F_1\left(\begin{matrix}-k, -k - \frac{1}{2}\\ -\lambda - 2k + r - \frac{1}{2}\end{matrix}; z_{\mathrm{a}}, {}^t z_{\mathrm{s}}^{-1}\right). $$

Motivation: Restriction to subgroups

- Consider the subgroups of $SU(r,r)$,

$$G_{\mathrm{s}} = Sp(r,\mathbb{R}) := \left\{ g \in SU(r,r) \;\middle|\; {}^t g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \right\},$$

$$G_{\mathrm{a}} = SO^*(2r) := \left\{ g \in SU(r,r) \;\middle|\; {}^t g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \right\}.$$

- $D_{\mathrm{s}} \subset \mathfrak{p}_{\mathrm{s}}^+ = \mathrm{Sym}(r,\mathbb{C})$, $D_{\mathrm{a}} \subset \mathfrak{p}_{\mathrm{a}}^+ = \mathrm{Alt}(r,\mathbb{C})$: Bounded symmetric domains.

$$D_\circ := \{x_\circ \in \mathfrak{p}_\circ^+ \mid I - x_\circ x_\circ^* \text{ is positive definite}\}.$$

- Let $\lambda \in \mathbb{C}$, (τ, W): a representation of $U(r)$.
Then $\widetilde{G}_\circ$ acts on $\mathcal{O}(D_\circ, W) = \mathcal{O}_\lambda(D_\circ, W)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(x) := \det(cx+d)^{-\frac{\lambda}{\delta}} \tau\left({}^t(cx+d)\right) f\left((ax+b)(cx+d)^{-1}\right),$$

where $\delta = 1$ for $\circ = \mathrm{s}$, $\delta = 2$ for $\circ = \mathrm{a}$.

- $\mathcal{H}_\lambda(D_\circ, W) \subset \mathcal{O}_\lambda(D_\circ, W)$: Unitary subrepresentation of $\widetilde{G}_\circ$ (if it exists).

Motivation: Restriction to subgroups

- Consider the subgroups of $SU(r,r)$,

$$G_{\mathrm{s}} = Sp(r,\mathbb{R}) := \left\{ g \in SU(r,r) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \right\},$$

$$G_{\mathrm{a}} = SO^*(2r) := \left\{ g \in SU(r,r) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \right\}.$$

- $D_{\mathrm{s}} \subset \mathfrak{p}_{\mathrm{s}}^+ = \mathrm{Sym}(r,\mathbb{C})$, $D_{\mathrm{a}} \subset \mathfrak{p}_{\mathrm{a}}^+ = \mathrm{Alt}(r,\mathbb{C})$: Bounded symmetric domains.

$$D_\circ := \{ x_\circ \in \mathfrak{p}_\circ^+ \mid I - x_\circ x_\circ^* \text{ is positive definite} \}.$$

- Let $\lambda \in \mathbb{C}$, (τ, W): a representation of $U(r)$.
  Then $\widetilde{G}_\circ$ acts on $\mathcal{O}(D_\circ, W) = \mathcal{O}_\lambda(D_\circ, W)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(x) := \det(cx+d)^{-\frac{\lambda}{\delta}} \tau\left({}^t(cx+d)\right) f\left((ax+b)(cx+d)^{-1}\right),$$

  where $\delta = 1$ for $\circ = \mathrm{s}$, $\delta = 2$ for $\circ = \mathrm{a}$.
- $\mathcal{H}_\lambda(D_\circ, W) \subset \mathcal{O}_\lambda(D_\circ, W)$: Unitary subrepresentation of $\widetilde{G}_\circ$ (if it exists).

Motivation: Restriction to subgroups

- Consider the subgroups of $SU(r,r)$,

$$G_{\mathrm{s}} = Sp(r,\mathbb{R}) := \left\{ g \in SU(r,r) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \right\},$$

$$G_{\mathrm{a}} = SO^*(2r) := \left\{ g \in SU(r,r) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \right\}.$$

- $D_{\mathrm{s}} \subset \mathfrak{p}_{\mathrm{s}}^+ = \mathrm{Sym}(r,\mathbb{C})$, $D_{\mathrm{a}} \subset \mathfrak{p}_{\mathrm{a}}^+ = \mathrm{Alt}(r,\mathbb{C})$: Bounded symmetric domains.

$$D_\circ := \{ x_\circ \in \mathfrak{p}_\circ^+ \mid I - x_\circ x_\circ^* \text{ is positive definite} \}.$$

- Let $\lambda \in \mathbb{C}$, (τ, W): a representation of $U(r)$.
  Then $\widetilde{G}_\circ$ acts on $\mathcal{O}(D_\circ, W) = \mathcal{O}_\lambda(D_\circ, W)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(x) := \det(cx+d)^{-\frac{\lambda}{\delta}} \tau\left({}^t(cx+d)\right) f\left((ax+b)(cx+d)^{-1}\right),$$

  where $\delta = 1$ for $\circ = \mathrm{s}$, $\delta = 2$ for $\circ = \mathrm{a}$.

- $\mathcal{H}_\lambda(D_\circ, W) \subset \mathcal{O}_\lambda(D_\circ, W)$: Unitary subrepresentation of $\widetilde{G}_\circ$ (if it exists).

Motivation: Restriction to subgroups

- Consider the subgroups of $SU(r,r)$,

$$G_{\mathrm{s}} = Sp(r,\mathbb{R}) := \left\{ g \in SU(r,r) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \right\},$$

$$G_{\mathrm{a}} = SO^*(2r) := \left\{ g \in SU(r,r) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \right\}.$$

- $D_{\mathrm{s}} \subset \mathfrak{p}^+_{\mathrm{s}} = \mathrm{Sym}(r,\mathbb{C})$, $D_{\mathrm{a}} \subset \mathfrak{p}^+_{\mathrm{a}} = \mathrm{Alt}(r,\mathbb{C})$: Bounded symmetric domains.

$$D_\circ := \{ x_\circ \in \mathfrak{p}^+_\circ \mid I - x_\circ x_\circ^* \text{ is positive definite} \}.$$

- Let $\lambda \in \mathbb{C}$, (τ, W): a representation of $U(r)$.
  Then $\widetilde{G}_\circ$ acts on $\mathcal{O}(D_\circ, W) = \mathcal{O}_\lambda(D_\circ, W)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(x) := \det(cx+d)^{-\frac{\lambda}{\delta}} \tau\left({}^t(cx+d)\right) f\left((ax+b)(cx+d)^{-1}\right),$$

  where $\delta = 1$ for $\circ = \mathrm{s}$, $\delta = 2$ for $\circ = \mathrm{a}$.
- $\mathcal{H}_\lambda(D_\circ, W) \subset \mathcal{O}_\lambda(D_\circ, W)$: Unitary subrepresentation of $\widetilde{G}_\circ$ (if it exists).

Motivation: Restriction to subgroups (conti.)

- (Kobayashi (2008)) According to the decompositions

$$\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^{+}) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{+}) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} V_{\mathbf{m}^2}^{\vee},$$

$$\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^{+}) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{r}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{s}}^{+}) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{r}} V_{2\mathbf{m}}^{\vee},$$

(where $\mathbf{m}^2 := (m_1, m_1, \ldots, m_{\lfloor r/2\rfloor}, m_{\lfloor r/2\rfloor}(,0)))$, for $\lambda > 2r-1$ we have

$$\mathcal{H}_\lambda(D)|_{\widetilde{Sp}(r,\mathbb{R})} \simeq \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}}^{\oplus} \mathcal{H}_\lambda(D_{\mathrm{s}}, V_{\mathbf{m}^2}^{\vee}),$$

$$\mathcal{H}_\lambda(D)|_{\widetilde{SO^*}(2r)} \simeq \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r}}^{\oplus} \mathcal{H}_{2\lambda}(D_{\mathrm{a}}, V_{2\mathbf{m}}^{\vee}).$$

- For $W \simeq V_{\mathbf{m}^2}^{\vee}$ or $W \simeq V_{2\mathbf{m}}^{\vee}$, want to construct explicitly the $\widetilde{G}_\circ$-intertwining operators (symmetry breaking operators)

$$\mathcal{F}_{\lambda,W}^{\circ} : \mathcal{H}_\lambda(D)|_{\widetilde{G}_\circ} \longrightarrow \mathcal{H}_{\delta\lambda}(D_\circ, W).$$

- Today: Consider $\mathbf{m} = (\underline{k+1}_l, \underline{k}_{s-l})$ or $\mathbf{m} = (k+l, \underline{k}_{r-1})$.

Motivation: Restriction to subgroups (conti.)

- (Kobayashi (2008)) According to the decompositions

$$\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^{+}) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^{+}) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} V_{\mathbf{m}^2}^{\vee},$$

$$\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^{+}) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{r}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{s}}^{+}) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{r}} V_{2\mathbf{m}}^{\vee},$$

(where $\mathbf{m}^2 := (m_1, m_1, \ldots, m_{\lfloor r/2\rfloor}, m_{\lfloor r/2\rfloor}(,0)))$, for $\lambda > 2r-1$ we have

$$\mathcal{H}_\lambda(D)|_{\widetilde{Sp}(r,\mathbb{R})} \simeq \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}}^{\oplus} \mathcal{H}_\lambda(D_{\mathrm{s}}, V_{\mathbf{m}^2}^{\vee}),$$

$$\mathcal{H}_\lambda(D)|_{\widetilde{SO^*}(2r)} \simeq \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r}}^{\oplus} \mathcal{H}_{2\lambda}(D_{\mathrm{a}}, V_{2\mathbf{m}}^{\vee}).$$

- For $W \simeq V_{\mathbf{m}^2}^{\vee}$ or $W \simeq V_{2\mathbf{m}}^{\vee}$, want to construct explicitly the $\widetilde{G}_\circ$-intertwining operators (symmetry breaking operators)

$$\mathcal{F}_{\lambda,W}^{\circ} \colon \mathcal{H}_\lambda(D)|_{\widetilde{G}_\circ} \longrightarrow \mathcal{H}_{\delta\lambda}(D_\circ, W).$$

- Today: Consider $\mathbf{m} = (\underline{k+1}_l, \underline{k}_{s-l})$ or $\mathbf{m} = (k+l, \underline{k}_{r-1})$.

Motivation: Restriction to subgroups (conti.)

- (Kobayashi (2008)) According to the decompositions

$$\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{a}}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}} V_{\mathbf{m}^2}^{\vee},$$

$$\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{r}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{s}}^+) \simeq \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^{r}} V_{2\mathbf{m}}^{\vee},$$

(where $\mathbf{m}^2 := (m_1, m_1, \ldots, m_{\lfloor r/2\rfloor}, m_{\lfloor r/2\rfloor}(,0)))$, for $\lambda > 2r-1$ we have

$$\mathcal{H}_\lambda(D)|_{\widetilde{Sp}(r,\mathbb{R})} \simeq \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor r/2\rfloor}}^{\oplus} \mathcal{H}_\lambda(D_{\mathrm{s}}, V_{\mathbf{m}^2}^{\vee}),$$

$$\mathcal{H}_\lambda(D)|_{\widetilde{SO^*}(2r)} \simeq \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r}}^{\oplus} \mathcal{H}_{2\lambda}(D_{\mathrm{a}}, V_{2\mathbf{m}}^{\vee}).$$

- For $W \simeq V_{\mathbf{m}^2}^{\vee}$ or $W \simeq V_{2\mathbf{m}}^{\vee}$, want to construct explicitly the $\widetilde{G}_\circ$-intertwining operators (symmetry breaking operators)

$$\mathcal{F}_{\lambda,W}^{\circ} \colon \mathcal{H}_\lambda(D)|_{\widetilde{G}_\circ} \longrightarrow \mathcal{H}_{\delta\lambda}(D_\circ, W).$$

- Today: Consider $\mathbf{m} = (\underline{k+1}_l, \underline{k}_{s-l})$ or $\mathbf{m} = (k+l, \underline{k}_{r-1})$.

Symmetry breaking operators for $Sp(2s, \mathbb{R})$

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k \underbrace{f(x_{\mathrm{a}})}_{\in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+)}, e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda, x} = C_{\mathrm{a}}^{2s}(\lambda, k, l) F_{\mathrm{a}}^{l} \left(\begin{matrix} -\frac{k}{2} \\ -\lambda - k + 2s \end{matrix}; f; z \right).$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{a}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^-) \otimes V^{\vee}_{(\underline{1}_{2l}, \underline{0}_{2s-2l})})^{U(2s)}$. *Then the linear map*

$$\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} \colon \mathcal{H}_\lambda(D_{SU(2s,2s)}) \longrightarrow \mathcal{H}_{\lambda+k}(D_{Sp(2s,\mathbb{R})}, V^{\vee}_{(\underline{1}_{2l}, \underline{0}_{2s-2l})}),$$

$$(\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} f)(x_{\mathrm{s}}) := F_{\mathrm{a}}^{l} \left(\begin{matrix} -\frac{k}{2} \\ -\lambda - k + 2s \end{matrix}; \mathrm{K}_l; \frac{\partial}{\partial x} \right) f(x) \bigg|_{x_{\mathrm{a}}=0}$$

intertwines the $\widetilde{Sp}(2s, \mathbb{R})$*-action, and its operator norm is given by*

$$\|\mathcal{F}^{\mathrm{s}}_{\lambda,k,l}\|^2_{\mathrm{op}} = \frac{C_{\mathrm{a},2s,k,l}}{C_{\mathrm{a}}^{2s}(\lambda, k, l)},$$

where $\exists C_{\mathrm{a},2s,k,l} > 0$ *is independent of* λ.

Symmetry breaking operators for $Sp(2s, \mathbb{R})$

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k \underbrace{f(x_{\mathrm{a}})}_{\in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+)}, e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{a}}^{2s}(\lambda,k,l) F_{\mathrm{a}}^{l}\left({-\frac{k}{2} \atop -\lambda - k + 2s}; f; z \right).$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{a}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^-) \otimes V^{\vee}_{(\underline{1}_{2l},\underline{0}_{2s-2l})})^{U(2s)}$*. Then the linear map*

$$\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} \colon \mathcal{H}_\lambda(D_{SU(2s,2s)}) \longrightarrow \mathcal{H}_{\lambda+k}(D_{Sp(2s,\mathbb{R})}, V^{\vee}_{(\underline{1}_{2l},\underline{0}_{2s-2l})}),$$

$$(\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} f)(x_{\mathrm{s}}) := F_{\mathrm{a}}^{l}\left({-\frac{k}{2} \atop -\lambda - k + 2s}; \mathrm{K}_l; \frac{\partial}{\partial x} \right) f(x)\bigg|_{x_{\mathrm{a}}=0}$$

intertwines the $\widetilde{Sp}(2s,\mathbb{R})$*-action, and its operator norm is given by*

$$\|\mathcal{F}^{\mathrm{s}}_{\lambda,k,l}\|^2_{\mathrm{op}} = \frac{C_{\mathrm{a},2s,k,l}}{C_{\mathrm{a}}^{2s}(\lambda,k,l)},$$

where $\exists C_{\mathrm{a},2s,k,l} > 0$ *is independent of* λ*.*

Symmetry breaking operators for $Sp(2s, \mathbb{R})$

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k \underbrace{f(x_{\mathrm{a}})}_{\in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+)}, e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{a}}^{2s}(\lambda,k,l) F_{\mathrm{a}}^l \left(\begin{matrix} -\frac{k}{2} \\ -\lambda - k + 2s \end{matrix}; f; z \right).$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{a}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^-) \otimes V^\vee_{(\underline{1}_{2l}, \underline{0}_{2s-2l})})^{U(2s)}$. *Then the linear map*

$$\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} \colon \mathcal{H}_\lambda(D_{SU(2s,2s)}) \longrightarrow \mathcal{H}_{\lambda+k}(D_{Sp(2s,\mathbb{R})}, V^\vee_{(\underline{1}_{2l}, \underline{0}_{2s-2l})}),$$

$$(\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} f)(x_{\mathrm{s}}) := F_{\mathrm{a}}^l \left(\begin{matrix} -\frac{k}{2} \\ -\lambda - k + 2s \end{matrix}; \mathrm{K}_l; \frac{\partial}{\partial x} \right) f(x) \bigg|_{x_{\mathrm{a}}=0}$$

intertwines the $\widetilde{Sp}(2s, \mathbb{R})$*-action, and its operator norm is given by*

$$\|\mathcal{F}^{\mathrm{s}}_{\lambda,k,l}\|^2_{\mathrm{op}} = \frac{C_{\mathrm{a},2s,k,l}}{C_{\mathrm{a}}^{2s}(\lambda,k,l)},$$

where $\exists C_{\mathrm{a},2s,k,l} > 0$ *is independent of* λ.

Symmetry breaking operators for $Sp(2s,\mathbb{R})$

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k \underbrace{f(x_{\mathrm{a}})}_{\in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+)}, e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C_{\mathrm{a}}^{2s}(\lambda,k,l) F_{\mathrm{a}}^{l}\left({-\frac{k}{2} \atop -\lambda-k+2s}; f; z \right).$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{a}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{a}}^-) \otimes V^{\vee}_{(\underline{1}_{2l},\underline{0}_{2s-2l})})^{U(2s)}$. *Then the linear map*

$$\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} \colon \mathcal{H}_\lambda(D_{SU(2s,2s)}) \longrightarrow \mathcal{H}_{\lambda+k}(D_{Sp(2s,\mathbb{R})}, V^{\vee}_{(\underline{1}_{2l},\underline{0}_{2s-2l})}),$$

$$(\mathcal{F}^{\mathrm{s}}_{\lambda,k,l} f)(x_{\mathrm{s}}) := F_{\mathrm{a}}^{l}\left({-\frac{k}{2} \atop -\lambda-k+2s}; \mathrm{K}_l; \frac{\partial}{\partial x} \right) f(x) \bigg|_{x_{\mathrm{a}}=0}$$

intertwines the $\widetilde{Sp}(2s,\mathbb{R})$*-action, and its operator norm is given by*

$$\|\mathcal{F}^{\mathrm{s}}_{\lambda,k,l}\|^2_{\mathrm{op}} = \frac{C_{\mathrm{a},2s,k,l}}{C_{\mathrm{a}}^{2s}(\lambda,k,l)},$$

where $\exists C_{\mathrm{a},2s,k,l} > 0$ *is independent of* λ.

Symmetry breaking operators for $SO^*(2r)$

$$\Big\langle \det(x_{\mathrm{s}})^k \underbrace{f(x_{\mathrm{s}})}_{\in\mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_{\mathrm{s}}^+)}, e^{\mathrm{tr}(xz^*)} \Big\rangle_{\lambda,x} = C_{\mathrm{s}}^r(\lambda,k,l) F_{\mathrm{s}}^l\left(\begin{matrix}-k\\ -\lambda-2k+r\end{matrix}; f; z\right),$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{s}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^-) \otimes V_{(2l,\underline{0}_{r-1})}^\vee)^{U(r)}$. *Then the linear map*

$$\mathcal{F}_{\lambda,k,l}^{\mathrm{a}} \colon \mathcal{H}_\lambda(D_{SU(r,r)}) \longrightarrow \mathcal{H}_{2\lambda+4k}(D_{SO^*(2r)}, V_{(2l,\underline{0}_{r-1})}^\vee),$$

$$(\mathcal{F}_{\lambda,k,l}^{\mathrm{a}} f)(x_{\mathrm{a}}) := F_{\mathrm{s}}^l\left(\begin{matrix}-k\\ -\lambda-2k+r\end{matrix}; \mathrm{K}_l; \frac{\partial}{\partial x}\right) f(x)\Big|_{x_{\mathrm{s}}=0}$$

intertwines the $\widetilde{SO^*}(2r)$*-action, and its operator norm is given by*

$$\|\mathcal{F}_{\lambda,k,l}^{\mathrm{a}}\|_{\mathrm{op}}^2 = \frac{C_{\mathrm{s},r,k,l}}{C_{\mathrm{s}}^r(\lambda,k,l)},$$

where $\exists C_{\mathrm{s},r,k,l} > 0$ *is independent of* λ.

Symmetry breaking operators for $SO^*(2r)$

$$\Big\langle \det(x_{\mathrm{s}})^k \underbrace{f(x_{\mathrm{s}})}_{\in\mathcal{P}_{(l,0,\dots,0)}(\mathfrak{p}_{\mathrm{s}}^+)}, e^{\mathrm{tr}(xz^*)} \Big\rangle_{\lambda,x} = C_{\mathrm{s}}^r(\lambda,k,l) F_{\mathrm{s}}^l \left(\begin{matrix} -k \\ -\lambda - 2k + r \end{matrix} ; f; z \right),$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{s}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^-) \otimes V^\vee_{(2l,\underline{0}_{r-1})})^{U(r)}$. *Then the linear map*

$$\mathcal{F}^{\mathrm{a}}_{\lambda,k,l} \colon \mathcal{H}_\lambda(D_{SU(r,r)}) \longrightarrow \mathcal{H}_{2\lambda+4k}(D_{SO^*(2r)}, V^\vee_{(2l,\underline{0}_{r-1})}),$$

$$(\mathcal{F}^{\mathrm{a}}_{\lambda,k,l} f)(x_{\mathrm{a}}) := F_{\mathrm{s}}^l \left(\begin{matrix} -k \\ -\lambda - 2k + r \end{matrix} ; \mathrm{K}_l; \frac{\partial}{\partial x} \right) f(x) \Big|_{x_{\mathrm{s}}=0}$$

intertwines the $\widetilde{SO}^*(2r)$*-action, and its operator norm is given by*

$$\|\mathcal{F}^{\mathrm{a}}_{\lambda,k,l}\|^2_{\mathrm{op}} = \frac{C_{\mathrm{s},r,k,l}}{C_{\mathrm{s}}^r(\lambda,k,l)},$$

where $\exists C_{\mathrm{s},r,k,l} > 0$ *is independent of* λ.

Symmetry breaking operators for $SO^*(2r)$

$$\Big\langle \det(x_{\mathrm{s}})^k \underbrace{f(x_{\mathrm{s}})}_{\in\mathcal{P}_{(l,0,\dots,0)}(\mathfrak{p}_{\mathrm{s}}^+)}, e^{\operatorname{tr}(xz^*)} \Big\rangle_{\lambda,x} = C_{\mathrm{s}}^r(\lambda,k,l) F_{\mathrm{s}}^l \left(\begin{matrix} -k \\ -\lambda-2k+r \end{matrix}; f; z \right),$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{s}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^-) \otimes V_{(2l,\underline{0}_{r-1})}^\vee)^{U(r)}$. *Then the linear map*

$$\mathcal{F}_{\lambda,k,l}^{\mathrm{a}} \colon \mathcal{H}_\lambda(D_{SU(r,r)}) \longrightarrow \mathcal{H}_{2\lambda+4k}(D_{SO^*(2r)}, V_{(2l,\underline{0}_{r-1})}^\vee),$$

$$(\mathcal{F}_{\lambda,k,l}^{\mathrm{a}} f)(x_{\mathrm{a}}) := F_{\mathrm{s}}^l \left(\begin{matrix} -k \\ -\lambda-2k+r \end{matrix}; \mathrm{K}_l; \frac{\partial}{\partial x} \right) f(x) \Big|_{x_{\mathrm{s}}=0}$$

intertwines the $\widetilde{SO}^*(2r)$*-action, and its operator norm is given by*

$$\|\mathcal{F}_{\lambda,k,l}^{\mathrm{a}}\|_{\mathrm{op}}^2 = \frac{C_{\mathrm{s},r,k,l}}{C_{\mathrm{s}}^r(\lambda,k,l)},$$

where $\exists C_{\mathrm{s},r,k,l} > 0$ *is independent of* λ.

Symmetry breaking operators for $SO^*(2r)$

$$\Big\langle \det(x_{\mathrm{s}})^k \underbrace{f(x_{\mathrm{s}})}_{\in\mathcal{P}_{(l,0,\dots,0)}(\mathfrak{p}_{\mathrm{s}}^+)}, e^{\mathrm{tr}(xz^*)} \Big\rangle_{\lambda,x} = C_{\mathrm{s}}^r(\lambda,k,l) F_{\mathrm{s}}^l \left(\begin{matrix} -k \\ -\lambda-2k+r \end{matrix} ; f ; z \right),$$

Corollary (N)

We take $\mathrm{K}_l(x_{\mathrm{s}}) \in (\mathcal{P}(\mathfrak{p}_{\mathrm{s}}^-) \otimes V_{(2l,\underline{0}_{r-1})}^\vee)^{U(r)}$. *Then the linear map*

$$\mathcal{F}_{\lambda,k,l}^{\mathrm{a}} \colon \mathcal{H}_\lambda(D_{SU(r,r)}) \longrightarrow \mathcal{H}_{2\lambda+4k}(D_{SO^*(2r)}, V_{(2l,\underline{0}_{r-1})}^\vee),$$

$$(\mathcal{F}_{\lambda,k,l}^{\mathrm{a}} f)(x_{\mathrm{a}}) := F_{\mathrm{s}}^l \left(\begin{matrix} -k \\ -\lambda-2k+r \end{matrix} ; \mathrm{K}_l ; \frac{\partial}{\partial x} \right) f(x) \Big|_{x_{\mathrm{s}}=0}$$

intertwines the $\widetilde{SO}^*(2r)$*-action, and its operator norm is given by*

$$\|\mathcal{F}_{\lambda,k,l}^{\mathrm{a}}\|_{\mathrm{op}}^2 = \frac{C_{\mathrm{s},r,k,l}}{C_{\mathrm{s}}^r(\lambda,k,l)},$$

where $\exists C_{\mathrm{s},r,k,l} > 0$ *is independent of* λ.

Previous works on differential SBO

Previous works on differential symmetry breaking operators:

- Rankin (1956), Cohen (1975): $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), SL(2,\mathbb{R}))$.
- Peng–Zhang (2004): $(G \times G, G)$ (G: Hermitian).
- Juhl (2009): $(SO_0(1,n), SO_0(1,n-1))$.
- Ibukiyama–Kuzumaki–Ochiai (2012): $(Sp(2n,\mathbb{R}), Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R}))$.
- Kobayashi–Ørsted–Somberg–Souček (2015): $(SO_0(p,q), SO_0(p,q-1))$.
- Kobayashi–Pevzner (2016): $(SO_0(2,n), SO_0(2,n-1))$, $(Sp(n,\mathbb{R}), Sp(n-1,\mathbb{R}) \times Sp(1,\mathbb{R}))$, $(U(n,1) \times U(n,1), U(n,1))$ etc.
- Kobayashi–Kubo–Pevzner (2016): $(O(1,n), O(1,n-1))$ (vector-valued).

Results for other symmetric pairs (arXiv:2105.13976)

- Computable for $\mathcal{P}_{(\underline{k+1}_l, \underline{k}_{s-l})}((\mathfrak{p}^+)^{-\sigma})$ for the following $(G, G^\sigma, \mathfrak{p}^+, (\mathfrak{p}^+)^{-\sigma})$.

$(Sp(2s,\mathbb{R}),$	$Sp(s,\mathbb{R}) \times Sp(s,\mathbb{R}),$	$\mathrm{Sym}(2s,\mathbb{C}),$	$M(s,\mathbb{C})),$
$(SU(2s,2s),$	$Sp(2s,\mathbb{R}),$	$M(2s,\mathbb{C}),$	$\mathrm{Alt}(2s,\mathbb{C})).$

- Computable for $\mathcal{P}_{(k+l, \underline{k}_{r-1})}((\mathfrak{p}^+)^{-\sigma})$ for the following $(G, G^\sigma, \mathfrak{p}^+, (\mathfrak{p}^+)^{-\sigma})$.

$(SO_0(2,n),$	$SO_0(2,n') \times SO(n''),$	$\mathbb{C}^n,$	$\mathbb{C}^{n''})$ $(n'' \neq 2),$
$(SO^*(4r),$	$SO^*(2r) \times SO^*(2r),$	$\mathrm{Alt}(2r,\mathbb{C}),$	$M(r,\mathbb{C})),$
$(SU(r,r),$	$SO^*(2r),$	$M(r,\mathbb{C}),$	$\mathrm{Sym}(r,\mathbb{C})),$
$(E_{7(-25)},$	$SU(2,6),$	$\mathrm{Herm}(3,\mathbb{O})^{\mathbb{C}},$	$\mathrm{Alt}(6,\mathbb{C})).$

- Computable for $\mathcal{P}_{\underline{k}_{r'}}((\mathfrak{p}^+)_1^{-\sigma}) \boxtimes \mathcal{P}_{\mathbf{l}}((\mathfrak{p}^+)_2^{-\sigma})$ for the fol. $(G, G^\sigma, \mathfrak{p}^+, (\mathfrak{p}^+)^{-\sigma})$.

$(SO_0(2,n),$	$SO_0(2,n-2) \times SO(2),$	$\mathbb{C}^n,$	$\mathbb{C} \oplus \mathbb{C}),$
$(Sp(r,\mathbb{R}),$	$U(r',r''),$	$\mathrm{Sym}(r,\mathbb{C}),$	$\mathrm{Sym}(r',\mathbb{C}) \oplus \mathrm{Sym}(r'',\mathbb{C})),$
$(U(r,r),$	$U(r',r'') \times U(r'',r'),$	$M(r,\mathbb{C}),$	$M(r',\mathbb{C}) \oplus M(r'',\mathbb{C})),$
$(SO^*(4r),$	$U(2r',2r''),$	$\mathrm{Alt}(2r,\mathbb{C}),$	$\mathrm{Alt}(2r',\mathbb{C}) \oplus \mathrm{Alt}(2r'',\mathbb{C})),$
$(E_{7(-25)},$	$U(1) \times E_{6(-14)},$	$\mathrm{Herm}(3,\mathbb{O})^{\mathbb{C}},$	$\mathbb{C} \oplus \mathrm{Herm}(2,\mathbb{O})^{\mathbb{C}}).$

Results for other symmetric pairs (arXiv:2105.13976)

- Computable for $\mathcal{P}_{(\underline{k+1}_l,\underline{k}_{s-l})}((\mathfrak{p}^+)^{-\sigma})$ for the following $(G, G^\sigma, \mathfrak{p}^+, (\mathfrak{p}^+)^{-\sigma})$.

$$\begin{array}{llll} (Sp(2s,\mathbb{R}), & Sp(s,\mathbb{R})\times Sp(s,\mathbb{R}), & \mathrm{Sym}(2s,\mathbb{C}), & M(s,\mathbb{C})), \\ (SU(2s,2s), & Sp(2s,\mathbb{R}), & M(2s,\mathbb{C}), & \mathrm{Alt}(2s,\mathbb{C})). \end{array}$$

- Computable for $\mathcal{P}_{(k+l,\underline{k}_{r-1})}((\mathfrak{p}^+)^{-\sigma})$ for the following $(G, G^\sigma, \mathfrak{p}^+, (\mathfrak{p}^+)^{-\sigma})$.

$$\begin{array}{llll} (SO_0(2,n), & SO_0(2,n')\times SO(n''), & \mathbb{C}^n, & \mathbb{C}^{n''}) \quad (n''\neq 2), \\ (SO^*(4r), & SO^*(2r)\times SO^*(2r), & \mathrm{Alt}(2r,\mathbb{C}), & M(r,\mathbb{C})), \\ (SU(r,r), & SO^*(2r), & M(r,\mathbb{C}), & \mathrm{Sym}(r,\mathbb{C})), \\ (E_{7(-25)}, & SU(2,6), & \mathrm{Herm}(3,\mathbb{O})^{\mathbb{C}}, & \mathrm{Alt}(6,\mathbb{C})). \end{array}$$

- Computable for $\mathcal{P}_{\underline{k}_{r'}}((\mathfrak{p}^+)_1^{-\sigma})\boxtimes\mathcal{P}_{\mathbf{l}}((\mathfrak{p}^+)_2^{-\sigma})$ for the fol. $(G, G^\sigma, \mathfrak{p}^+, (\mathfrak{p}^+)^{-\sigma})$.

$$\begin{array}{llll} (SO_0(2,n), & SO_0(2,n-2)\times SO(2), & \mathbb{C}^n, & \mathbb{C}\oplus\mathbb{C}), \\ (Sp(r,\mathbb{R}), & U(r',r''), & \mathrm{Sym}(r,\mathbb{C}), & \mathrm{Sym}(r',\mathbb{C})\oplus\mathrm{Sym}(r'',\mathbb{C})), \\ (U(r,r), & U(r',r'')\times U(r'',r'), & M(r,\mathbb{C}), & M(r',\mathbb{C})\oplus M(r'',\mathbb{C})), \\ (SO^*(4r), & U(2r',2r''), & \mathrm{Alt}(2r,\mathbb{C}), & \mathrm{Alt}(2r',\mathbb{C})\oplus\mathrm{Alt}(2r'',\mathbb{C})), \\ (E_{7(-25)}, & U(1)\times E_{6(-14)}, & \mathrm{Herm}(3,\mathbb{O})^{\mathbb{C}}, & \mathbb{C}\oplus\mathrm{Herm}(2,\mathbb{O})^{\mathbb{C}}). \end{array}$$

Outline

Main theorem

$\mathfrak{p}^{\pm} := M(r, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{s}} := \mathrm{Sym}(r, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{a}} := \mathrm{Alt}(r, \mathbb{C})$.

Theorem (N)

Let $k \in \mathbb{Z}_{\geq 0}$, and let

$$\begin{array}{lllll} r = 2s, & 0 \leq l < s, & f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^{+}_{\mathrm{a}}), & \varepsilon = 2 & ((\circ, \bullet) = (\mathrm{s}, \mathrm{a})), \\ r : \text{general}, & l \in \mathbb{Z}_{\geq 0}, & f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^{+}_{\mathrm{s}}), & \varepsilon = 1 & ((\circ, \bullet) = (\mathrm{a}, \mathrm{s})). \end{array}$$

Then for $\mathrm{Re}\,\lambda > 2r - 1$ *and for* $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \mathfrak{p}^{+}$, *we have*

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C^r_\bullet(\lambda, k, l) F^l_\bullet \left(\begin{matrix} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{matrix} ; f; z \right).$$

The proof is divided into two parts.

$$(A) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \exists C F^l_\bullet \left(\begin{matrix} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{matrix} ; f; z \right),$$

$$(B) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ = 0} = C^r_\bullet(\lambda, k, l) \det(z_\bullet)^{k/\varepsilon} f(z_\bullet).$$

Main theorem

$\mathfrak{p}^{\pm} := M(r,\mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{s}} := \mathrm{Sym}(r,\mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{a}} := \mathrm{Alt}(r,\mathbb{C})$.

Theorem (N)

Let $k \in \mathbb{Z}_{\geq 0}$, and let

$$\begin{array}{llll l} r = 2s, & 0 \leq l < s, & f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}}), & \varepsilon = 2 & ((\circ,\bullet) = (\mathrm{s},\mathrm{a})), \\ r: \text{ general}, & l \in \mathbb{Z}_{\geq 0}, & f(x_{\mathrm{s}}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_{\mathrm{s}}), & \varepsilon = 1 & ((\circ,\bullet) = (\mathrm{a},\mathrm{s})). \end{array}$$

Then for $\mathrm{Re}\,\lambda > 2r-1$ *and for* $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \mathfrak{p}^+$*, we have*

$$\left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = C^r_\bullet(\lambda,k,l) F^l_\bullet \left(\begin{array}{c} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{array} ; f; z \right).$$

The proof is divided into two parts.

$$(A) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \exists C F^l_\bullet \left(\begin{array}{c} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{array} ; f; z \right),$$

$$(B) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ = 0} = C^r_\bullet(\lambda,k,l) \det(z_\bullet)^{k/\varepsilon} f(z_\bullet).$$

F-method

- (τ, W): an irreducible representation of $K^\sigma := U(r)$.
- $\widetilde{G} := \widetilde{SU}(r,r) \curvearrowright \mathcal{O}_\lambda(D)$, $\widetilde{G}_{\mathrm{s}} := \widetilde{Sp}(r,\mathbb{R}) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{s}}, W)$,
  $\widetilde{G}_{\mathrm{a}} := \widetilde{SO^*}(2r) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{a}}, W)$.

Theorem (F-method, Kobayashi–Pevzner (2016))

Let $(\circ, \delta) \in \{(\mathrm{s}, 1), (\mathrm{a}, 2)\}$. The symbol map gives the isomorphism

$$\mathrm{Hom}_{\widetilde{G}_\circ}(\mathcal{O}_\lambda(D), \mathcal{O}_{\delta\lambda}(D_\circ, W)) \simeq \left((\mathcal{P}(\mathfrak{p}^-) \cap \mathrm{Sol}(\mathcal{B}^-_{\lambda,\circ})) \otimes W\right)^{U(r)},$$

where $\mathcal{B}^\pm_{\lambda,\circ} : \mathcal{O}(\mathfrak{p}^\pm) \longrightarrow \mathcal{O}(\mathfrak{p}^\pm) \otimes \mathfrak{p}^\mp_\circ$ is an explicit differential operator.

$\lambda > 2r-1 \Longrightarrow$ The above is 1-dimensional.

Corollary

Let $(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$. For $\lambda > 2r-1$, $\mathbf{k} \in \mathbb{Z}^{\lfloor r/\varepsilon \rfloor}_{++}$, we have

$$\mathrm{Hom}_{U(r)}\left(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_\bullet), \mathcal{P}(\mathfrak{p}^+) \cap \mathrm{Sol}(\mathcal{B}^+_{\lambda,\circ})\right) = \mathbb{C}\left(F(x_\bullet) \mapsto \left\langle F(x_\bullet), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}\right).$$

F-method

- (τ, W): an irreducible representation of $K^\sigma := U(r)$.
- $\widetilde{G} := \widetilde{SU}(r,r) \curvearrowright \mathcal{O}_\lambda(D)$, $\widetilde{G}_{\mathrm{s}} := \widetilde{Sp}(r,\mathbb{R}) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{s}}, W)$,
  $\widetilde{G}_{\mathrm{a}} := \widetilde{SO^*}(2r) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{a}}, W)$.

Theorem (F-method, Kobayashi–Pevzner (2016))

Let $(\circ, \delta) \in \{(\mathrm{s}, 1), (\mathrm{a}, 2)\}$. *The symbol map gives the isomorphism*

$$\mathrm{Hom}_{\widetilde{G}_\circ}(\mathcal{O}_\lambda(D), \mathcal{O}_{\delta\lambda}(D_\circ, W)) \simeq \left(\left(\mathcal{P}(\mathfrak{p}^-) \cap \mathrm{Sol}(\mathcal{B}^-_{\lambda,\circ})\right) \otimes W\right)^{U(r)},$$

where $\mathcal{B}^\pm_{\lambda,\circ} \colon \mathcal{O}(\mathfrak{p}^\pm) \longrightarrow \mathcal{O}(\mathfrak{p}^\pm) \otimes \mathfrak{p}^\mp_\circ$ *is an explicit differential operator.*

$\lambda > 2r - 1 \implies$ The above is 1-dimensional.

Corollary

Let $(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$. *For* $\lambda > 2r-1$, $\mathbf{k} \in \mathbb{Z}^{\lfloor r/\varepsilon \rfloor}_{++}$, *we have*

$$\mathrm{Hom}_{U(r)}\left(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_\bullet), \mathcal{P}(\mathfrak{p}^+) \cap \mathrm{Sol}(\mathcal{B}^+_{\lambda,\circ})\right) = \mathbb{C}\left(F(x_\bullet) \mapsto \left\langle F(x_\bullet), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}\right).$$

F-method

- (τ, W): an irreducible representation of $K^\sigma := U(r)$.
- $\widetilde{G} := \widetilde{SU}(r,r) \curvearrowright \mathcal{O}_\lambda(D)$, $\widetilde{G}_{\mathrm{s}} := \widetilde{Sp}(r,\mathbb{R}) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{s}}, W)$, $\widetilde{G}_{\mathrm{a}} := \widetilde{SO^*}(2r) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{a}}, W)$.

Theorem (F-method, Kobayashi–Pevzner (2016))

Let $(\circ, \delta) \in \{(\mathrm{s}, 1), (\mathrm{a}, 2)\}$. The symbol map gives the isomorphism

$$\mathrm{Hom}_{\widetilde{G}_\circ}(\mathcal{O}_\lambda(D), \mathcal{O}_{\delta\lambda}(D_\circ, W)) \simeq \left(\left(\mathcal{P}(\mathfrak{p}^-) \cap \mathrm{Sol}(\mathcal{B}^-_{\lambda,\circ})\right) \otimes W\right)^{U(r)},$$

where $\mathcal{B}^\pm_{\lambda,\circ} : \mathcal{O}(\mathfrak{p}^\pm) \longrightarrow \mathcal{O}(\mathfrak{p}^\pm) \otimes \mathfrak{p}^\mp_\circ$ is an explicit differential operator.

$\lambda > 2r - 1 \Longrightarrow$ The above is 1-dimensional.

Corollary

Let $(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$. For $\lambda > 2r - 1$, $\mathbf{k} \in \mathbb{Z}^{\lfloor r/\varepsilon \rfloor}_{++}$, we have

$$\mathrm{Hom}_{U(r)}\left(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_\bullet), \mathcal{P}(\mathfrak{p}^+) \cap \mathrm{Sol}(\mathcal{B}^+_{\lambda,\circ})\right) = \mathbb{C}\left(F(x_\bullet) \mapsto \left\langle F(x_\bullet), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}\right).$$

F-method

- (τ, W): an irreducible representation of $K^\sigma := U(r)$.
- $\widetilde{G} := \widetilde{SU}(r,r) \curvearrowright \mathcal{O}_\lambda(D)$, $\widetilde{G}_{\mathrm{s}} := \widetilde{Sp}(r,\mathbb{R}) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{s}}, W)$,
  $\widetilde{G}_{\mathrm{a}} := \widetilde{SO^*}(2r) \curvearrowright \mathcal{O}_\lambda(D_{\mathrm{a}}, W)$.

Theorem (F-method, Kobayashi–Pevzner (2016))

Let $(\circ, \delta) \in \{(\mathrm{s}, 1), (\mathrm{a}, 2)\}$. *The symbol map gives the isomorphism*

$$\mathrm{Hom}_{\widetilde{G}_\circ}(\mathcal{O}_\lambda(D), \mathcal{O}_{\delta\lambda}(D_\circ, W)) \simeq \left(\left(\mathcal{P}(\mathfrak{p}^-) \cap \mathrm{Sol}(\mathcal{B}^-_{\lambda,\circ})\right) \otimes W\right)^{U(r)},$$

where $\mathcal{B}^\pm_{\lambda,\circ} \colon \mathcal{O}(\mathfrak{p}^\pm) \longrightarrow \mathcal{O}(\mathfrak{p}^\pm) \otimes \mathfrak{p}^\mp_\circ$ *is an explicit differential operator.*

$\lambda > 2r - 1 \Longrightarrow$ The above is 1-dimensional.

Corollary

Let $(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$. *For* $\lambda > 2r - 1$, $\mathbf{k} \in \mathbb{Z}^{\lfloor r/\varepsilon \rfloor}_{++}$, *we have*

$$\mathrm{Hom}_{U(r)}\left(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_\bullet), \mathcal{P}(\mathfrak{p}^+) \cap \mathrm{Sol}(\mathcal{B}^+_{\lambda,\circ})\right) = \mathbb{C}\left(F(x_\bullet) \mapsto \left\langle F(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}\right).$$

How to prove (A)

Corollary

Let $(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$. *For* $\lambda > 2r - 1$, $\mathbf{k} \in \mathbb{Z}_{++}^{\lfloor r/\varepsilon \rfloor}$, *we have*

$$\mathrm{Hom}_{U(r)}\left(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_\bullet^+), \mathcal{P}(\mathfrak{p}^+) \cap \mathrm{Sol}(\mathcal{B}_{\lambda,\circ}^+)\right) = \mathbb{C}\left(F(x_\bullet) \mapsto \left\langle F(x_\bullet), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}\right).$$

For the proof of

$$(A) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x} = \exists C F_\bullet^l\left(\begin{matrix} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{matrix}; f; z\right),$$

enough to show, for $f(x_\mathrm{a}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_\mathrm{a}^+)$ or $f(x_\mathrm{s}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_\mathrm{s}^+)$,

$$(A1) \quad F_\bullet^l\left(\begin{matrix}\nu \\ \mu\end{matrix}; f(g(\cdot)^t g); z\right) = \det(g)^{\varepsilon\nu} F_\bullet^l\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; gz^t g\right) \quad (g \in GL(r, \mathbb{C})),$$

$$(A2) \quad \mathcal{B}_{-\mu+2\nu+2s,\circ}^+ F_\bullet^l\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; z\right) = 0.$$

These are proved by using an integral expression for $F_\bullet^l$.

How to prove (A)

Corollary

Let $(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$. *For* $\lambda > 2r - 1$, $\mathbf{k} \in \mathbb{Z}_{++}^{\lfloor r/\varepsilon \rfloor}$, *we have*

$$\mathrm{Hom}_{U(r)}\left(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_\bullet^+), \mathcal{P}(\mathfrak{p}^+) \cap \mathrm{Sol}(\mathcal{B}_{\lambda,\circ}^+)\right) = \mathbb{C}\left(F(x_\bullet) \mapsto \left\langle F(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}\right).$$

For the proof of

$$(A) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \exists C F_\bullet^l \left(\begin{matrix} -\frac{k}{\varepsilon} \\ -\lambda - \frac{2k}{\varepsilon} + r \end{matrix}; f; z \right),$$

enough to show, for $f(x_\mathrm{a}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_\mathrm{a}^+)$ or $f(x_\mathrm{s}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_\mathrm{s}^+)$,

$$(A1) \quad F_\bullet^l \left(\begin{matrix} \nu \\ \mu \end{matrix}; f(g(\cdot)^t g); z \right) = \det(g)^{\varepsilon\nu} F_\bullet^l \left(\begin{matrix} \nu \\ \mu \end{matrix}; f; gz^t g \right) \quad (g \in GL(r, \mathbb{C})),$$

$$(A2) \quad \mathcal{B}_{-\mu+2\nu+2s,\circ}^+ F_\bullet^l \left(\begin{matrix} \nu \\ \mu \end{matrix}; f; z \right) = 0.$$

These are proved by using an integral expression for $F_\bullet^l$.

How to prove (B)

$(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$.
For the proof of

$$(B) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ=0} = C_\bullet^r(\lambda, k, l) \det(z_\bullet)^{k/\varepsilon} f(z_\bullet),$$

we prove, for $f(x_\mathrm{a}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_\mathrm{a}^+)$ or $f(x_\mathrm{s}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_\mathrm{s}^+)$,

$$(B') \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ=0}$$
$$= \begin{cases} \dfrac{\det(z_\mathrm{a})^{-\lambda+r}}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \mathrm{Pf}\left(\dfrac{\partial}{\partial z_\mathrm{a}}\right)^k \det(z_\mathrm{a})^{\lambda+k-r} f(z_\mathrm{a}) & (\bullet = \mathrm{a}) \\ \dfrac{\det(z_\mathrm{s})^{-\lambda+r}}{(\lambda)_{(2k+l, \underline{2k}_{r-1})}} \det\left(\dfrac{\partial}{\partial z_\mathrm{s}}\right)^k \det(z_\mathrm{s})^{\lambda+2k-r} f(z_\mathrm{s}) & (\bullet = \mathrm{s}), \end{cases}$$

by rewriting the inner product by the inverse Laplace transform (Gindikin (1964) + Faraut–Korányi (1990)).

How to prove (B)

$(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$.
For the proof of

$$(B) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ=0} = C^r_\bullet(\lambda, k, l) \det(z_\bullet)^{k/\varepsilon} f(z_\bullet),$$

we prove, for $f(x_\mathrm{a}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_\mathrm{a})$ or $f(x_\mathrm{s}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}^+_\mathrm{s})$,

$$(B') \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ=0}$$
$$= \begin{cases} \dfrac{\det(z_\mathrm{a})^{-\lambda+r}}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \operatorname{Pf}\left(\dfrac{\partial}{\partial z_\mathrm{a}}\right)^k \det(z_\mathrm{a})^{\lambda+k-r} f(z_\mathrm{a}) & (\bullet = \mathrm{a}) \\ \dfrac{\det(z_\mathrm{s})^{-\lambda+r}}{(\lambda)_{(2k+l, \underline{2k}_{r-1})}} \det\left(\dfrac{\partial}{\partial z_\mathrm{s}}\right)^k \det(z_\mathrm{s})^{\lambda+2k-r} f(z_\mathrm{s}) & (\bullet = \mathrm{s}), \end{cases}$$

by rewriting the inner product by the inverse Laplace transform (Gindikin (1964) + Faraut–Korányi (1990)).

How to prove (B)

$(\circ, \bullet, \varepsilon) \in \{(\mathrm{s}, \mathrm{a}, 2), (\mathrm{a}, \mathrm{s}, 1)\}$.
For the proof of

$$(B) \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ=0} = C_\bullet^r(\lambda,k,l) \det(z_\bullet)^{k/\varepsilon} f(z_\bullet),$$

we prove, for $f(x_\mathrm{a}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_\mathrm{a}^+)$ or $f(x_\mathrm{s}) \in \mathcal{P}_{(l,0,\ldots,0)}(\mathfrak{p}_\mathrm{s}^+)$,

$$(B') \quad \left\langle \det(x_\bullet)^{k/\varepsilon} f(x_\bullet), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} \Big|_{z_\circ=0}$$
$$= \begin{cases} \dfrac{\det(z_\mathrm{a})^{-\lambda+r}}{(\lambda)_{(\underline{k+1}_l,\underline{k}_{r-l})}} \mathrm{Pf}\left(\dfrac{\partial}{\partial z_\mathrm{a}}\right)^k \det(z_\mathrm{a})^{\lambda+k-r} f(z_\mathrm{a}) & (\bullet = \mathrm{a}) \\ \dfrac{\det(z_\mathrm{s})^{-\lambda+r}}{(\lambda)_{(2k+l,\underline{2k}_{r-1})}} \det\left(\dfrac{\partial}{\partial z_\mathrm{s}}\right)^k \det(z_\mathrm{s})^{\lambda+2k-r} f(z_\mathrm{s}) & (\bullet = \mathrm{s}), \end{cases}$$

by rewriting the inner product by the inverse Laplace transform (Gindikin (1964) + Faraut–Korányi (1990)).

What to prove

Suppose $(\circ, \bullet) = (\mathrm{s}, \mathrm{a})$, $r = 2s$. Enough to show, for $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+)$,

$$(A1) \quad F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f(g(\cdot)^t g); z \Bigg) = \det(g)^{2\nu} F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f; gz^t g \Bigg) \quad (g \in GL(r, \mathbb{C})).$$

$$(A2) \quad \mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s}} F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f; z \Bigg) = 0.$$

$\Longleftarrow$ Use an integral expression for F_{a}^l.

$$(B') \quad \left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{\det(z)^{-\lambda+r}}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \det(z)^{\lambda+k-r} f(z_{\mathrm{a}}).$$

$\Longleftarrow$ Rewrite the inner product by the inverse Laplace transform
(Gindikin (1964) + Faraut–Korányi (1990)).

What to prove

Suppose $(\circ,\bullet)=(\mathrm{s},\mathrm{a})$, $r=2s$. Enough to show, for $f(x_{\mathrm{a}})\in\mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+)$,

$$(A1)\quad F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix};f(g(\cdot)^t g);z\right)=\det(g)^{2\nu}F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix};f;gz^t g\right)\quad(g\in GL(r,\mathbb{C})).$$

$$(A2)\quad \mathcal{B}^{+}_{-\mu+2\nu+2s,\mathrm{s}}F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix};f;z\right)=0.$$

$\Longleftarrow$ Use an integral expression for F_{a}^{l}.

$$(B')\quad \left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}),e^{\mathrm{tr}(xz^*)}\right\rangle_{\lambda,x}=\frac{\det(z)^{-\lambda+r}}{(\lambda)_{(\underline{k+1}_l,\underline{k}_{r-l})}}\mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^{k}\det(z)^{\lambda+k-r}f(z_{\mathrm{a}}).$$

$\Longleftarrow$ Rewrite the inner product by the inverse Laplace transform (Gindikin (1964) + Faraut–Korányi (1990)).

Laplace transform, inverse Laplace transform

- Fix $J \in SU(r) \subset M(r, \mathbb{C}) =: \mathfrak{p}^+$.
- $\Omega := \mathrm{Herm}_+(r, \mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^+ := \mathrm{Herm}(r, \mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^+ := M(r, \mathbb{C})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$\Gamma_r(\lambda) := (2\pi)^{r(r-1)/2} \prod_{j=1}^{r} \Gamma(\lambda - (j-1)), \quad (\lambda)_{\mathbf{m}} := \prod_{j=1}^{r} (\lambda - (j-1))_{m_j}.$$

Proposition (Gindikin (1964))

Let $\mathrm{Re}\,\lambda > 2r - 1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$, $f \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$. *For* $w \in \Omega + \sqrt{-1}\mathfrak{n}^+$, $z, a \in \Omega$, *we have*

$$\int_\Omega e^{-\mathrm{tr}(zJ^* wJ^*)} f(z) \det(z)^{\lambda - r} dz = \Gamma_r(\lambda)(\lambda)_{\mathbf{m}} f(Jw^{-1}J) \det(w)^{-\lambda},$$

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^* wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z) \det(z)^{\lambda - r}.$$

Laplace transform, inverse Laplace transform

- Fix $J \in SU(r) \subset M(r,\mathbb{C}) =: \mathfrak{p}^+$.
- $\Omega := \mathrm{Herm}_+(r,\mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^+ := \mathrm{Herm}(r,\mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^+ := M(r,\mathbb{C})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$\Gamma_r(\lambda) := (2\pi)^{r(r-1)/2} \prod_{j=1}^{r} \Gamma(\lambda - (j-1)), \quad (\lambda)_{\mathbf{m}} := \prod_{j=1}^{r} (\lambda - (j-1))_{m_j}.$$

Proposition (Gindikin (1964))

Let $\mathrm{Re}\,\lambda > 2r-1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$, $f \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$. *For* $w \in \Omega + \sqrt{-1}\mathfrak{n}^+$, $z, a \in \Omega$, *we have*

$$\int_\Omega e^{-\mathrm{tr}(zJ^*wJ^*)} f(z) \det(z)^{\lambda-r} dz = \Gamma_r(\lambda)(\lambda)_{\mathbf{m}} f(Jw^{-1}J) \det(w)^{-\lambda},$$

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z) \det(z)^{\lambda-r}.$$

Laplace transform, inverse Laplace transform

- Fix $J \in SU(r) \subset M(r,\mathbb{C}) =: \mathfrak{p}^+$.
- $\Omega := \mathrm{Herm}_+(r,\mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^+ := \mathrm{Herm}(r,\mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^+ := M(r,\mathbb{C})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$\Gamma_r(\lambda) := (2\pi)^{r(r-1)/2} \prod_{j=1}^{r} \Gamma(\lambda - (j-1)), \quad (\lambda)_{\mathbf{m}} := \prod_{j=1}^{r} (\lambda - (j-1))_{m_j}.$$

Proposition (Gindikin (1964))

Let $\mathrm{Re}\,\lambda > 2r-1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$, $f \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$. *For* $w \in \Omega + \sqrt{-1}\mathfrak{n}^+$, $z, a \in \Omega$, *we have*

$$\int_\Omega e^{-\mathrm{tr}(zJ^*wJ^*)} f(z) \det(z)^{\lambda-r} dz = \Gamma_r(\lambda)(\lambda)_{\mathbf{m}} f(Jw^{-1}J) \det(w)^{-\lambda},$$

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z) \det(z)^{\lambda-r}.$$

Laplace transform, inverse Laplace transform

- Fix $J \in SU(r) \subset M(r,\mathbb{C}) =: \mathfrak{p}^+$.
- $\Omega := \mathrm{Herm}_+(r,\mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^+ := \mathrm{Herm}(r,\mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^+ := M(r,\mathbb{C})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$\Gamma_r(\lambda) := (2\pi)^{r(r-1)/2} \prod_{j=1}^{r} \Gamma(\lambda - (j-1)), \quad (\lambda)_{\mathbf{m}} := \prod_{j=1}^{r} (\lambda - (j-1))_{m_j}.$$

Proposition (Gindikin (1964))

Let $\mathrm{Re}\,\lambda > 2r-1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$, $f \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$. *For* $w \in \Omega + \sqrt{-1}\mathfrak{n}^+$, $z, a \in \Omega$, *we have*

$$\int_\Omega e^{-\operatorname{tr}(zJ^*wJ^*)} f(z) \det(z)^{\lambda-r} dz = \Gamma_r(\lambda)(\lambda)_{\mathbf{m}} f(Jw^{-1}J) \det(w)^{-\lambda},$$

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z) \det(z)^{\lambda-r}.$$

Laplace transform, inverse Laplace transform

- Fix $J \in SU(r) \subset M(r, \mathbb{C}) =: \mathfrak{p}^+$.
- $\Omega := \mathrm{Herm}_+(r, \mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^+ := \mathrm{Herm}(r, \mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^+ := M(r, \mathbb{C})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$, let

$$\Gamma_r(\lambda) := (2\pi)^{r(r-1)/2} \prod_{j=1}^{r} \Gamma(\lambda - (j-1)), \quad (\lambda)_{\mathbf{m}} := \prod_{j=1}^{r} (\lambda - (j-1))_{m_j}.$$

Proposition (Gindikin (1964))

Let $\mathrm{Re}\,\lambda > 2r - 1$, $\mathbf{m} \in \mathbb{Z}^r_{++}$, $f \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$. *For* $w \in \Omega + \sqrt{-1}\mathfrak{n}^+$, $z, a \in \Omega$, *we have*

$$\int_\Omega e^{-\mathrm{tr}(zJ^* wJ^*)} f(z) \det(z)^{\lambda - r} dz = \Gamma_r(\lambda)(\lambda)_{\mathbf{m}} f(Jw^{-1}J) \det(w)^{-\lambda},$$

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^* wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z) \det(z)^{\lambda - r}.$$

Inverse Laplace transform and Bergman inner product

For $\operatorname{Re}\lambda > 2r-1$, $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ we have

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z) \det(z)^{\lambda-r}.$$

Comparing this with Faraut–Korányi's result

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_{\mathbf{m}}} f(z),$$

we get

Corollary

For $\operatorname{Re}\lambda > 2r-1$, $f \in \mathcal{P}(\mathfrak{p}^+)$, $z, a \in \Omega$ *we have*

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw.$$

Suppose $r = 2s$, $J := \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \in SU(r) \cap \mathfrak{p}^+_{\mathrm{a}}$.

Inverse Laplace transform and Bergman inner product

For $\operatorname{Re}\lambda > 2r-1$, $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ we have

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z) \det(z)^{\lambda-r}.$$

Comparing this with Faraut–Korányi's result

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_{\mathbf{m}}} f(z),$$

we get

Corollary

For $\operatorname{Re}\lambda > 2r-1$, $f \in \mathcal{P}(\mathfrak{p}^+)$, $z, a \in \Omega$ *we have*

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)} f(Jw^{-1}J) \det(w)^{-\lambda} dw.$$

Suppose $r = 2s$, $J := \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \in SU(r) \cap \mathfrak{p}^+_{\mathrm{a}}$.

Inverse Laplace transform and Bergman inner product

For $\operatorname{Re}\lambda > 2r-1$, $f(x)\in\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ we have

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}}\int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)}f(Jw^{-1}J)\det(w)^{-\lambda}dw = \frac{1}{(\lambda)_{\mathbf{m}}}f(z)\det(z)^{\lambda-r}.$$

Comparing this with Faraut–Korányi's result

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)}\right\rangle_{\lambda,x} = \frac{1}{(\lambda)_{\mathbf{m}}}f(z),$$

we get

Corollary

For $\operatorname{Re}\lambda > 2r-1$, $f\in\mathcal{P}(\mathfrak{p}^+)$, $z,a\in\Omega$ *we have*

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)}\right\rangle_{\lambda,x} = \det(z)^{-\lambda+r}\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}}\int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)}f(Jw^{-1}J)\det(w)^{-\lambda}dw.$$

Suppose $r=2s$, $J:=\begin{pmatrix}0 & I_s\\ -I_s & 0\end{pmatrix}\in SU(r)\cap\mathfrak{p}^+_{\mathrm{a}}$.

Inverse Laplace transform and Bergman inner product

For $\operatorname{Re}\lambda > 2r-1$, $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ we have

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)} f(Jw^{-1}J)\det(w)^{-\lambda}dw = \frac{1}{(\lambda)_{\mathbf{m}}} f(z)\det(z)^{\lambda-r}.$$

Comparing this with Faraut–Korányi's result

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_{\mathbf{m}}} f(z),$$

we get

Corollary

For $\operatorname{Re}\lambda > 2r-1$, $f \in \mathcal{P}(\mathfrak{p}^+)$, $z, a \in \Omega$ *we have*

$$\left\langle f(x), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\operatorname{tr}(zJ^*wJ^*)} f(Jw^{-1}J)\det(w)^{-\lambda}dw.$$

Suppose $r = 2s$, $J := \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \in SU(r) \cap \mathfrak{p}^+_{\mathrm{a}}$.

Proof of (B')

Hence for $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}}) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(x))^k f(\mathrm{Proj}_{\mathrm{a}}(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J))^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_{\mathrm{a}}{}^t w^{-1}{}^t J) = \det(w)^{-1}\mathrm{Pf}(w_{\mathrm{a}})\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_{\mathrm{a}})^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_{\mathrm{a}})$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \det(z)^{\lambda+k-r} f(z_{\mathrm{a}}) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}}) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(x))^k f(\mathrm{Proj}_{\mathrm{a}}(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J))^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_{\mathrm{a}}{}^t w^{-1}{}^t J) = \det(w)^{-1}\,\mathrm{Pf}(w_{\mathrm{a}})\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_{\mathrm{a}})^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_{\mathrm{a}})$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \det(z)^{\lambda+k-r} f(z_{\mathrm{a}}) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_s + z_a \in \Omega$, $f(x_a) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_a^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_a)^k f(x_a), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_a(x))^k f(\mathrm{Proj}_a(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J))^k f(\mathrm{Proj}_a(Jw^{-1}J))$$

$$\times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_a{}^t w^{-1\,t}J) = \det(w)^{-1}\mathrm{Pf}(w_a)\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_a)^k f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_a)$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l,\underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \det(z)^{\lambda+k-r} f(z_a) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_s + z_a \in \Omega$, $f(x_a) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_a^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_a)^k f(x_a), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_a(x))^k f(\mathrm{Proj}_a(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J))^k f(\mathrm{Proj}_a(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$(\mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_a{}^t w^{-1}{}^t J) = \det(w)^{-1}\,\mathrm{Pf}(w_a))$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_a)^k f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_a)$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \det(z)^{\lambda+k-r} f(z_a) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_s + z_a \in \Omega$, $f(x_a) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_a^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_a)^k f(x_a), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_a(x))^k f(\mathrm{Proj}_a(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^* wJ^*)} \mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J))^k f(\mathrm{Proj}_a(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1} w_a {}^t w^{-1} {}^t J) = \det(w)^{-1} \mathrm{Pf}(w_a)\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^* wJ^*)} \mathrm{Pf}(w_a)^k f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^* wJ^*)} f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_a)$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \det(z)^{\lambda+k-r} f(z_a) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}}) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(x))^k f(\mathrm{Proj}_{\mathrm{a}}(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J))^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_{\mathrm{a}}{}^t w^{-1}{}^t J) = \det(w)^{-1}\,\mathrm{Pf}(w_{\mathrm{a}})\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_{\mathrm{a}})^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_{\mathrm{a}})$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l,\underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \det(z)^{\lambda+k-r} f(z_{\mathrm{a}}) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_s + z_a \in \Omega$, $f(x_a) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_a^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_a)^k f(x_a), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_a(x))^k f(\mathrm{Proj}_a(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J))^k f(\mathrm{Proj}_a(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_a {}^t w^{-1} {}^t J) = \det(w)^{-1} \mathrm{Pf}(w_a)\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_a)^k f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_a)$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \det(z)^{\lambda+k-r} f(z_a) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_s + z_a \in \Omega$, $f(x_a) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_a^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_a)^k f(x_a), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_a(x))^k f(\mathrm{Proj}_a(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J))^k f(\mathrm{Proj}_a(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_a {}^t w^{-1} {}^t J) = \det(w)^{-1} \mathrm{Pf}(w_a)\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_a)^k f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_a)$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \det(z)^{\lambda+k-r} f(z_a) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_s + z_a \in \Omega$, $f(x_a) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_a^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_a)^k f(x_a), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_a(x))^k f(\mathrm{Proj}_a(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J))^k f(\mathrm{Proj}_a(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_a{}^t w^{-1}{}^t J) = \det(w)^{-1}\mathrm{Pf}(w_a)\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_a)^k f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_a)$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \det(z)^{\lambda+k-r} f(z_a) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}}) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(x))^k f(\mathrm{Proj}_{\mathrm{a}}(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J))^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J))$$

$$\times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1} w_{\mathrm{a}} {}^t w^{-1} {}^t J) = \det(w)^{-1} \mathrm{Pf}(w_{\mathrm{a}})\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_{\mathrm{a}})^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_{\mathrm{a}})$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \det(z)^{\lambda+k-r} f(z_{\mathrm{a}}) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_s + z_a \in \Omega$, $f(x_a) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_a^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_a)^k f(x_a), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_a(x))^k f(\mathrm{Proj}_a(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J))^k f(\mathrm{Proj}_a(Jw^{-1}J))$$

$$\times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_a(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_a {}^t w^{-1} {}^t J) = \det(w)^{-1} \mathrm{Pf}(w_a)\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_a)^k f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_a(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_a)$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_a}\right)^k \det(z)^{\lambda+k-r} f(z_a) \qquad (r = 2s).$$

Proof of (B')

Hence for $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, we have

$$\left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \left\langle \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(x))^k f(\mathrm{Proj}_{\mathrm{a}}(x)), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x}$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J))^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \times \det(w)^{-\lambda} dw$$

$$\left(\mathrm{Pf}(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) = \mathrm{Pf}(Jw^{-1}w_{\mathrm{a}}{}^t w^{-1}{}^t J) = \det(w)^{-1}\mathrm{Pf}(w_{\mathrm{a}})\right)$$

$$= \det(z)^{-\lambda+r} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} \mathrm{Pf}(w_{\mathrm{a}})^k f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{\det(z)^{-\lambda+r}}{(\lambda)_{\underline{k}_r}} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^{r^2}} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zJ^*wJ^*)} f(\mathrm{Proj}_{\mathrm{a}}(Jw^{-1}J)) \det(w)^{-\lambda-k} dw$$

$$= \frac{1}{(\lambda)_{\underline{k}_r}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \frac{1}{(\lambda+k)_{\underline{1}_l}} \det(z)^{\lambda+k-r} f(z_{\mathrm{a}})$$

$$= \frac{1}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \det(z)^{-\lambda+r} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \det(z)^{\lambda+k-r} f(z_{\mathrm{a}}) \qquad (r = 2s).$$

What to prove

Suppose $(\circ, \bullet) = (\mathrm{s}, \mathrm{a})$, $r = 2s$. Enough to show, for $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+)$,

$$(A1) \quad F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f(g(\cdot)^t g); z \Big) = \det(g)^{2\nu} F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f; gz^t g \Big) \quad (g \in GL(r, \mathbb{C})).$$

$$(A2) \quad \mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s}} F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f; z \Big) = 0.$$

$\Longleftarrow$ Use an integral expression for F_{a}^l.

$$(B') \quad \left\langle \mathrm{Pf}(x_{\mathrm{a}})^k f(x_{\mathrm{a}}), e^{\mathrm{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{\det(z)^{-\lambda+r}}{(\lambda)_{(\underline{k+1}_l, \underline{k}_{r-l})}} \mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^k \det(z)^{\lambda+k-r} f(z_{\mathrm{a}}).$$

$\Longleftarrow$ Rewrite the inner product by the inverse Laplace transform (Gindikin (1964) + Faraut–Korányi (1990)).

What to prove

Suppose $(\circ,\bullet)=(\mathrm{s},\mathrm{a})$, $r=2s$. Enough to show, for $f(x_{\mathrm{a}})\in\mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^{+})$,

$$(A1)\quad F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix};f(g(\cdot)^{t}g);z\right)=\det(g)^{2\nu}F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix};f;gz^{t}g\right)\quad(g\in GL(r,\mathbb{C})).$$

$$(A2)\quad \mathcal{B}^{+}_{-\mu+2\nu+2s,\mathrm{s}}F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu\\ \mu\end{matrix};f;z\right)=0.$$

$\Longleftarrow$ Use an integral expression for F_{a}^{l}.

$$(B')\quad \left\langle \mathrm{Pf}(x_{\mathrm{a}})^{k}f(x_{\mathrm{a}}),e^{\mathrm{tr}(xz^{*})}\right\rangle_{\lambda,x}=\frac{\det(z)^{-\lambda+r}}{(\lambda)_{(\underline{k+1}_l,\underline{k}_{r-l})}}\mathrm{Pf}\left(\frac{\partial}{\partial z_{\mathrm{a}}}\right)^{k}\det(z)^{\lambda+k-r}f(z_{\mathrm{a}}).$$

$\Longleftarrow$ Rewrite the inner product by the inverse Laplace transform (Gindikin (1964) + Faraut–Korányi (1990)).

Notations

- $\mathfrak{p}^\pm := M(2s, \mathbb{C})$, $\mathfrak{p}^\pm_{\mathrm{s}} := \mathrm{Sym}(2s, \mathbb{C})$, $\mathfrak{p}^\pm_{\mathrm{a}} := \mathrm{Alt}(2s, \mathbb{C})$.
- $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathfrak{p}^\pm$.
- $\Omega := \mathrm{Herm}_+(2s, \mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^+ := \mathrm{Herm}(2s, \mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^+ := M(2s, \mathbb{C})$.
- $\Omega_{\mathrm{a}} := \Omega \cap \mathfrak{p}^+_{\mathrm{a}} \overset{\text{open cone}}{\subset} \mathfrak{n}^+_{\mathrm{a}} := \mathfrak{n}^+ \cap \mathfrak{p}^+_{\mathrm{a}} \overset{\text{real form}}{\subset} \mathfrak{p}^+_{\mathrm{a}} := \mathrm{Alt}(2s, \mathbb{C})$
  $\Longrightarrow \Omega_{\mathrm{a}} \simeq \mathrm{Herm}_+(s, \mathbb{H})$, $\mathfrak{n}^+_{\mathrm{a}} \simeq \mathrm{Herm}(s, \mathbb{H})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{n} \in \mathbb{C}^s$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^s$,

$$(\lambda + \mathbf{n})_{\mathbf{m}} := \prod_{j=1}^{s} (\lambda + n_j - (j-1))_{m_j},$$

$$(\lambda + \mathbf{n})_{\mathbf{m},\mathrm{a}} := \prod_{j=1}^{s} (\lambda + n_j - 2(j-1))_{m_j},$$

$$\Gamma^{\mathrm{a}}_s(\lambda + \mathbf{n}) := (2\pi)^{s(s-1)} \prod_{j=1}^{s} \Gamma(\lambda + n_j - 2(j-1)).$$

Notations

- $\mathfrak{p}^{\pm} := M(2s, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{s}} := \mathrm{Sym}(2s, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{a}} := \mathrm{Alt}(2s, \mathbb{C})$.
- $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathfrak{p}^{\pm}$.
- $\Omega := \mathrm{Herm}_+(2s, \mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^+ := \mathrm{Herm}(2s, \mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^+ := M(2s, \mathbb{C})$.
- $\Omega_{\mathrm{a}} := \Omega \cap \mathfrak{p}^+_{\mathrm{a}} \overset{\text{open cone}}{\subset} \mathfrak{n}^+_{\mathrm{a}} := \mathfrak{n}^+ \cap \mathfrak{p}^+_{\mathrm{a}} \overset{\text{real form}}{\subset} \mathfrak{p}^+_{\mathrm{a}} := \mathrm{Alt}(2s, \mathbb{C})$
  $\Longrightarrow \Omega_{\mathrm{a}} \simeq \mathrm{Herm}_+(s, \mathbb{H})$, $\quad \mathfrak{n}^+_{\mathrm{a}} \simeq \mathrm{Herm}(s, \mathbb{H})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{n} \in \mathbb{C}^s$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^s$,

$$(\lambda + \mathbf{n})_{\mathbf{m}} := \prod_{j=1}^{s} (\lambda + n_j - (j-1))_{m_j},$$

$$(\lambda + \mathbf{n})_{\mathbf{m},\mathrm{a}} := \prod_{j=1}^{s} (\lambda + n_j - 2(j-1))_{m_j},$$

$$\Gamma^{\mathrm{a}}_s(\lambda + \mathbf{n}) := (2\pi)^{s(s-1)} \prod_{j=1}^{s} \Gamma(\lambda + n_j - 2(j-1)).$$

Notations

- $\mathfrak{p}^{\pm} := M(2s, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{s}} := \mathrm{Sym}(2s, \mathbb{C})$, $\mathfrak{p}^{\pm}_{\mathrm{a}} := \mathrm{Alt}(2s, \mathbb{C})$.
- $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathfrak{p}^{\pm}$.
- $\Omega := \mathrm{Herm}_{+}(2s, \mathbb{C})J \overset{\text{open cone}}{\subset} \mathfrak{n}^{+} := \mathrm{Herm}(2s, \mathbb{C})J \overset{\text{real form}}{\subset} \mathfrak{p}^{+} := M(2s, \mathbb{C})$.
- $\Omega_{\mathrm{a}} := \Omega \cap \mathfrak{p}^{+}_{\mathrm{a}} \overset{\text{open cone}}{\subset} \mathfrak{n}^{+}_{\mathrm{a}} := \mathfrak{n}^{+} \cap \mathfrak{p}^{+}_{\mathrm{a}} \overset{\text{real form}}{\subset} \mathfrak{p}^{+}_{\mathrm{a}} := \mathrm{Alt}(2s, \mathbb{C})$
  $\Longrightarrow \Omega_{\mathrm{a}} \simeq \mathrm{Herm}_{+}(s, \mathbb{H})$, $\mathfrak{n}^{+}_{\mathrm{a}} \simeq \mathrm{Herm}(s, \mathbb{H})$.
- For $\lambda \in \mathbb{C}$, $\mathbf{n} \in \mathbb{C}^{s}$, $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^{s}$,

$$(\lambda + \mathbf{n})_{\mathbf{m}} := \prod_{j=1}^{s} (\lambda + n_j - (j-1))_{m_j},$$

$$(\lambda + \mathbf{n})_{\mathbf{m},\mathrm{a}} := \prod_{j=1}^{s} (\lambda + n_j - 2(j-1))_{m_j},$$

$$\Gamma^{\mathrm{a}}_{s}(\lambda + \mathbf{n}) := (2\pi)^{s(s-1)} \prod_{j=1}^{s} \Gamma(\lambda + n_j - 2(j-1)).$$

Integral expression of F^l_{a}

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}})$, $\mu, \nu \in \mathbb{C}$, $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega \subset \mathfrak{p}^+$,

$$F^l_{\mathrm{a}}\begin{pmatrix}\nu\\ \mu\end{pmatrix}; f; z\Big) = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s\\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}}\left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})}^{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^+_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}^-_{\mathrm{a}}) \,\ni}.$$

Express this by the Laplace and inverse Laplace transforms.

Proposition

For $\mathrm{Re}\,\mu > 2s - \frac{3}{2}$, $\mathrm{Re}\,\nu > s-1$ *and for* $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $a_{\mathrm{a}} \in \Omega_{\mathrm{a}}$ *with* $z_{\mathrm{s}} + a_{\mathrm{a}} \in \Omega$, *we have*

$$F^l_{\mathrm{a}}\begin{pmatrix}\nu\\ \mu\end{pmatrix}; f; z\Big) = \frac{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} f(x_{\mathrm{a}})\, dx_{\mathrm{a}}\right) dy_{\mathrm{a}}.$$

Integral expression of F^l_{a}

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}})$, $\mu, \nu \in \mathbb{C}$, $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega \subset \mathfrak{p}^+$,

$$F^l_{\mathrm{a}}\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right)$$

$$= \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s\\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})}^{\substack{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^+_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}^-_{\mathrm{a}})\\ \cup}}.$$

Express this by the Laplace and inverse Laplace transforms.

Proposition

For $\mathrm{Re}\,\mu > 2s - \frac{3}{2}$, $\mathrm{Re}\,\nu > s - 1$ *and for* $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $a_{\mathrm{a}} \in \Omega_{\mathrm{a}}$ *with* $z_{\mathrm{s}} + a_{\mathrm{a}} \in \Omega$, *we have*

$$F^l_{\mathrm{a}}\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right) = \frac{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)} \mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} f(x_{\mathrm{a}})\, dx_{\mathrm{a}} \right) dy_{\mathrm{a}}.$$

Integral expression of F^l_{a}

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}})$, $\mu, \nu \in \mathbb{C}$, $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega \subset \mathfrak{p}^+$,

$$F^l_{\mathrm{a}}\binom{\nu}{\mu}; f; z\Big) = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})}^{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^+_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}^-_{\mathrm{a}}) \ \cup}.$$

Express this by the Laplace and inverse Laplace transforms.

Proposition

For $\mathrm{Re}\,\mu > 2s - \frac{3}{2}$, $\mathrm{Re}\,\nu > s - 1$ *and for* $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $a_{\mathrm{a}} \in \Omega_{\mathrm{a}}$ *with* $z_{\mathrm{s}} + a_{\mathrm{a}} \in \Omega$, *we have*

$$F^l_{\mathrm{a}}\binom{\nu}{\mu}; f; z\Big) = \frac{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\mathrm{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)} \mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\mathrm{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} f(x_{\mathrm{a}})\, dx_{\mathrm{a}} \right) dy_{\mathrm{a}}.$$

Integral expression of F^l_{a}

For $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}})$, $\mu, \nu \in \mathbb{C}$, $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega \subset \mathfrak{p}^+$,

$$F^l_{\mathrm{a}}\begin{pmatrix}\nu\\ \mu\end{pmatrix}; f; z\Bigr) = \mathrm{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s\\ |\mathbf{l}|=l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})}^{\substack{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^+_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}^-_{\mathrm{a}})\\ \cup}}.$$

Express this by the Laplace and inverse Laplace transforms.

Proposition

For $\operatorname{Re}\mu > 2s - \frac{3}{2}$, $\operatorname{Re}\nu > s - 1$ *and for* $z = z_{\mathrm{s}} + z_{\mathrm{a}} \in \Omega$, $a_{\mathrm{a}} \in \Omega_{\mathrm{a}}$ *with* $z_{\mathrm{s}} + a_{\mathrm{a}} \in \Omega$, *we have*

$$F^l_{\mathrm{a}}\begin{pmatrix}\nu\\ \mu\end{pmatrix}; f; z\Bigr) = \frac{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)} \mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} f(x_{\mathrm{a}})\, dx_{\mathrm{a}}\right) dy_{\mathrm{a}}.$$

Proof for integral expression of F^l_{a}

Since

$$\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}}) = \det(x_{\mathrm{a}})^{-\mu}\det(I-z_{\mathrm{s}}x_{\mathrm{a}}^{-1}z_{\mathrm{s}}x_{\mathrm{a}}^{-1})^{-\mu/2}f(x_{\mathrm{a}})$$
$$= \mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}\sum_{\mathbf{m}\in\mathbb{Z}^s_{++}}\sum_{\substack{\mathbf{l}\in\{0,1\}^s\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}},{}^tx_{\mathrm{a}}^{-1})$$

holds, we have

$$\frac{\Gamma^{\mathrm{a}}_s(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\int_{\Omega_{\mathrm{a}}}e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^*z_{\mathrm{a}}J^*)}\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}}e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^*y_{\mathrm{a}}J^*)}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$
$$=\frac{\Gamma^{\mathrm{a}}_s(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\sum_{\mathbf{m}\in\mathbb{Z}^s_{++}}\sum_{\substack{\mathbf{l}\in\{0,1\}^s\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}\int_{\Omega_{\mathrm{a}}}e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^*z_{\mathrm{a}}J^*)}\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}}e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^*y_{\mathrm{a}}J^*)}\mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}},{}^tx_{\mathrm{a}}^{-1})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$

Proof for integral expression of F^l_{a}

Since

$$\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}}) = \det(x_{\mathrm{a}})^{-\mu}\det(I - z_{\mathrm{s}}x_{\mathrm{a}}^{-1}z_{\mathrm{s}}x_{\mathrm{a}}^{-1})^{-\mu/2}f(x_{\mathrm{a}})$$
$$= \mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}\sum_{\mathbf{m}\in\mathbb{Z}^s_{++}}\sum_{\substack{\mathbf{l}\in\{0,1\}^s\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t x_{\mathrm{a}}^{-1})$$

holds, we have

$$\frac{\Gamma^{\mathrm{a}}_s(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}}J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^* y_{\mathrm{a}}J^*)}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$
$$=\frac{\Gamma^{\mathrm{a}}_s(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\sum_{\mathbf{m}\in\mathbb{Z}^s_{++}}\sum_{\substack{\mathbf{l}\in\{0,1\}^s\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}}J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^* y_{\mathrm{a}}J^*)}\,\mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t x_{\mathrm{a}}^{-1})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$

Proof for integral expression of F^l_{a}

Since

$$\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}}) = \det(x_{\mathrm{a}})^{-\mu}\det(I-z_{\mathrm{s}}x_{\mathrm{a}}^{-1}z_{\mathrm{s}}x_{\mathrm{a}}^{-1})^{-\mu/2}f(x_{\mathrm{a}})$$
$$= \mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}\sum_{\mathbf{m}\in\mathbb{Z}^s_{++}}\sum_{\substack{\mathbf{l}\in\{0,1\}^s\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}},{}^t x_{\mathrm{a}}^{-1})$$

holds, we have

$$\frac{\Gamma^{\mathrm{a}}_s(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^*z_{\mathrm{a}}J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^*y_{\mathrm{a}}J^*)}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$
$$=\frac{\Gamma^{\mathrm{a}}_s(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\sum_{\mathbf{m}\in\mathbb{Z}^s_{++}}\sum_{\substack{\mathbf{l}\in\{0,1\}^s\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^*z_{\mathrm{a}}J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^*y_{\mathrm{a}}J^*)}\,\mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}},{}^t x_{\mathrm{a}}^{-1})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$

Proof for integral expression of F_{a}^{l}

Since

$$\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}}) = \det(x_{\mathrm{a}})^{-\mu}\det(I - z_{\mathrm{s}}x_{\mathrm{a}}^{-1}z_{\mathrm{s}}x_{\mathrm{a}}^{-1})^{-\mu/2}f(x_{\mathrm{a}})$$
$$= \mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{s}}\sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\ |\mathbf{l}|=l}}(\mu)_{\mathbf{m}}F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}},{}^{t}x_{\mathrm{a}}^{-1})$$

holds, we have

$$\frac{\Gamma_{s}^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_{l}))}{\Gamma_{s}^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_{l}))}\int_{\Omega_{\mathrm{a}}}e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^{*}z_{\mathrm{a}}J^{*})}\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}}e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^{*}y_{\mathrm{a}}J^{*})}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$
$$=\frac{\Gamma_{s}^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_{l}))}{\Gamma_{s}^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_{l}))}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{s}}\sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\ |\mathbf{l}|=l}}(\mu)_{\mathbf{m}}\int_{\Omega_{\mathrm{a}}}e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^{*}z_{\mathrm{a}}J^{*})}\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}}e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^{*}y_{\mathrm{a}}J^{*})}\mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}},{}^{t}x_{\mathrm{a}}^{-1})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$

Proof for integral expression of F_{a}^{l}

Since

$$\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}}) = \det(x_{\mathrm{a}})^{-\mu}\det(I - z_{\mathrm{s}}x_{\mathrm{a}}^{-1}z_{\mathrm{s}}x_{\mathrm{a}}^{-1})^{-\mu/2}f(x_{\mathrm{a}})$$
$$= \mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{s}}\sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}},{}^{t}x_{\mathrm{a}}^{-1})$$

holds, we have

$$\frac{\Gamma_s^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^*z_{\mathrm{a}}J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^*y_{\mathrm{a}}J^*)}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$
$$=\frac{\Gamma_s^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_l))}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{s}}\sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\|\mathbf{l}|=l}}(\mu)_{\mathbf{m}}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^*z_{\mathrm{a}}J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^*y_{\mathrm{a}}J^*)}\mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}},{}^{t}x_{\mathrm{a}}^{-1})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$

Proof for integral expression of F_{a}^{l}

Since

$$\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}}) = \det(x_{\mathrm{a}})^{-\mu}\det(I - z_{\mathrm{s}}x_{\mathrm{a}}^{-1}z_{\mathrm{s}}x_{\mathrm{a}}^{-1})^{-\mu/2}f(x_{\mathrm{a}})$$
$$= \mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{s}}\sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\ |\mathbf{l}|=l}}(\mu)_{\mathbf{m}}F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^{t}x_{\mathrm{a}}^{-1})$$

holds, we have

$$\frac{\Gamma_{s}^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_{l}))}{\Gamma_{s}^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_{l}))}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^{*}z_{\mathrm{a}}J^{*})}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^{*}y_{\mathrm{a}}J^{*})}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$
$$=\frac{\Gamma_{s}^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_{l}))}{\Gamma_{s}^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_{l}))}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{s}}\sum_{\substack{\mathbf{l}\in\{0,1\}^{s}\\ |\mathbf{l}|=l}}(\mu)_{\mathbf{m}}\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^{*}z_{\mathrm{a}}J^{*})}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}}\int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^{*}y_{\mathrm{a}}J^{*})}\,\mathrm{Pf}(x_{\mathrm{a}})^{-2\mu}F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^{t}x_{\mathrm{a}}^{-1})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$

Proof for integral expression of F^l_{a} (conti.)

$$= \frac{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} (\mu)_{\mathbf{m}} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^* y_{\mathrm{a}} J^*)} \operatorname{Pf}(x_{\mathrm{a}})^{-2\mu} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t x_{\mathrm{a}}^{-1})}^{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^+_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}^-_{\mathrm{a}}) \ni} dx_{\mathrm{a}} \right) dy_{\mathrm{a}}$$

$$= \frac{1}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}$$

$$\times \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)} \operatorname{Pf}(y_{\mathrm{a}})^{2\mu-(2s-1)} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, \overline{J}{}^t y_{\mathrm{a}} \overline{J})\, dy_{\mathrm{a}}$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m},\mathbf{l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}) = F^l_{\mathrm{a}}\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; z\right)$$

Proof for integral expression of F^{l}_{a} (conti.)

$$= \frac{\Gamma^{\mathrm{a}}_{s}(2\mu - (\underline{0}_{s-l}, \underline{1}_{l}))}{\Gamma^{\mathrm{a}}_{s}(2\nu - (\underline{0}_{s-l}, \underline{1}_{l}))} \sum_{\mathbf{m}\in\mathbb{Z}^{s}_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^{s} \\ |\mathbf{l}|=l}} (\mu)_{\mathbf{m}} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^{*}z_{\mathrm{a}}J^{*})} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^{+}_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^{*}y_{\mathrm{a}}J^{*})} \operatorname{Pf}(x_{\mathrm{a}})^{-2\mu} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^{t}x_{\mathrm{a}}^{-1})}^{\mathcal{P}_{\mathbf{m}^{2}}(\mathfrak{p}^{+}_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}^{-}_{\mathrm{a}}) \ \ni} \, dx_{\mathrm{a}} \right) dy_{\mathrm{a}}$$

$$= \frac{1}{\Gamma^{\mathrm{a}}_{s}(2\nu - (\underline{0}_{s-l}, \underline{1}_{l}))} \sum_{\mathbf{m}\in\mathbb{Z}^{s}_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^{s} \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_{l}))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l}),\mathrm{a}}}$$

$$\times \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^{*}z_{\mathrm{a}}J^{*})} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)} \operatorname{Pf}(y_{\mathrm{a}})^{2\mu-(2s-1)} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, \overline{J}{}^{t}y_{\mathrm{a}}\overline{J}) \, dy_{\mathrm{a}}$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^{s}_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^{s} \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}(2\nu - (\underline{0}_{s-l}, \underline{1}_{l}))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l}),\mathrm{a}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_{l}))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l}),\mathrm{a}}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^{t}z_{\mathrm{a}}^{-1})$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m},\mathbf{l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l})}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_{l}\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_{l})}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^{t}z_{\mathrm{a}}^{-1}) = F^{l}_{\mathrm{a}}\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; z\right)$$

Proof for integral expression of F_{a}^{l} (conti.)

$$= \frac{\Gamma_s^{\mathrm{a}}(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} (\mu)_{\mathbf{m}} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^+} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^* y_{\mathrm{a}} J^*)} \operatorname{Pf}(x_{\mathrm{a}})^{-2\mu} \overbrace{F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t x_{\mathrm{a}}^{-1})}^{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^+)\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}_{\mathrm{a}}^-) \ \cup} dx_{\mathrm{a}} \right) dy_{\mathrm{a}}$$

$$= \frac{1}{\Gamma_s^{\mathrm{a}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}$$

$$\times \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)} \operatorname{Pf}(y_{\mathrm{a}})^{2\mu-(2s-1)} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, \overline{J}{}^t y_{\mathrm{a}} \overline{J})\, dy_{\mathrm{a}}$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}} (2\nu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m},\mathbf{l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + (\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + (\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}) = F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; z\right)$$

Proof for integral expression of F^l_{a} (conti.)

$$= \frac{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} (\mu)_{\mathbf{m}} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^* y_{\mathrm{a}} J^*)} \operatorname{Pf}(x_{\mathrm{a}})^{-2\mu} \overbrace{F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t x_{\mathrm{a}}^{-1})}^{\substack{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^+_{\mathrm{s}})\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}^-_{\mathrm{a}}) \\ \cup}} dx_{\mathrm{a}} \right) dy_{\mathrm{a}}$$

$$= \frac{1}{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}$$

$$\times \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)} \operatorname{Pf}(y_{\mathrm{a}})^{2\mu-(2s-1)} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, \overline{J}{}^t y_{\mathrm{a}} \overline{J})\, dy_{\mathrm{a}}$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}^s_{++}} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}} (2\nu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m},\mathbf{l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F^{\mathrm{a}}_{\mathbf{m},\mathbf{l}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}) = F^l_{\mathrm{a}}\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; z\right)$$

Proof for integral expression of F_{a}^{l} (conti.)

$$= \frac{\Gamma_s^{\mathrm{a}}(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} (\mu)_{\mathbf{m}} \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\frac{1}{(2\pi\sqrt{-1})^{s(2s-1)}} \int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^+} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^* y_{\mathrm{a}} J^*)} \operatorname{Pf}(x_{\mathrm{a}})^{-2\mu} \overbrace{F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t x_{\mathrm{a}}^{-1})}^{\mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}_{\mathrm{s}}^+)\otimes\mathcal{P}_{2\mathbf{m}-\mathbf{l}}(\mathfrak{p}_{\mathrm{a}}^-)\ \Cup} dx_{\mathrm{a}} \right) dy_{\mathrm{a}}$$

$$= \frac{1}{\Gamma_s^{\mathrm{a}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}$$

$$\times \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)} \operatorname{Pf}(y_{\mathrm{a}})^{2\mu-(2s-1)} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, \overline{J}{}^t y_{\mathrm{a}} \overline{J})\, dy_{\mathrm{a}}$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^s} \sum_{\substack{\mathbf{l}\in\{0,1\}^s \\ |\mathbf{l}|=l}} \frac{(\mu)_{\mathbf{m}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}}{(2\mu - (\underline{0}_{s-l}, \underline{1}_l))_{2\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l),\mathrm{a}}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1})$$

$$= \operatorname{Pf}(z_{\mathrm{a}})^{-2\nu} \sum_{\mathbf{m},\mathbf{l}} \frac{(\nu)_{\mathbf{m}} \left(\nu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}}{\left(\mu + \left(\underline{\tfrac{1}{2}}_{s-l}, \underline{-\tfrac{1}{2}}_l\right)\right)_{\mathbf{m}-\mathbf{l}+(\underline{0}_{s-l},\underline{1}_l)}} F_{\mathbf{m},\mathbf{l}}^{\mathrm{a}}[f](z_{\mathrm{s}}, {}^t z_{\mathrm{a}}^{-1}) = F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; z\right)$$

Proof of (A2)

$$\mathcal{B}^+_{\lambda,\mathrm{s}} \colon \mathcal{O}(\mathfrak{p}^+) \longrightarrow \mathcal{O}(\mathfrak{p}^+) \otimes \mathfrak{p}^-_{\mathrm{s}},$$

$$(\mathcal{B}^+_{\lambda,\mathrm{s}} f)(z) := \mathrm{Proj}_{\mathfrak{p}^- \to \mathfrak{p}^-_{\mathrm{s}}} \left(\frac{1}{2} \sum_{\alpha,\beta} (e_\alpha {}^t z e_\beta + e_\beta {}^t z e_\alpha) \frac{\partial^2 f}{\partial z_\alpha \partial z_\beta}(z) + \lambda \sum_\alpha e_\alpha \frac{\partial f}{\partial z_\alpha}(z) \right),$$

where $\{e_\alpha\}$: a basis of $\mathfrak{p}^-$, $\{z_\alpha\}$: the coordinate of $\mathfrak{p}^+$.
Integrating by parts, we can show

$$(2\pi\sqrt{-1})^{s(2s-1)} \frac{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))} \mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z} F^l_{\mathrm{a}} \left(\begin{matrix} \nu \\ \mu \end{matrix} ; f; z \right)$$

$$= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)} \mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\int_{\mathfrak{a}_{\mathrm{a}} + \sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \mathcal{B}^+_{\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} f(x_{\mathrm{a}})\, dx_{\mathrm{a}} \right) dy_{\mathrm{a}}$$

$$= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J z_{\mathrm{a}} J)} \mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\int_{\mathfrak{a}_{\mathrm{a}} + \sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} \mathcal{B}^+_{-\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}} f(x_{\mathrm{a}})\, dx_{\mathrm{a}} \right) dy_{\mathrm{a}}.$$

Proof of (A2)

$$\mathcal{B}^+_{\lambda,\mathrm{s}} \colon \mathcal{O}(\mathfrak{p}^+) \longrightarrow \mathcal{O}(\mathfrak{p}^+) \otimes \mathfrak{p}_{\mathrm{s}}^-,$$

$$(\mathcal{B}^+_{\lambda,\mathrm{s}} f)(z) := \mathrm{Proj}_{\mathfrak{p}^- \to \mathfrak{p}_{\mathrm{s}}^-}\left(\frac{1}{2}\sum_{\alpha,\beta}(e_\alpha{}^t z e_\beta + e_\beta{}^t z e_\alpha)\frac{\partial^2 f}{\partial z_\alpha \partial z_\beta}(z) + \lambda \sum_\alpha e_\alpha \frac{\partial f}{\partial z_\alpha}(z)\right),$$

where $\{e_\alpha\}$: a basis of $\mathfrak{p}^-$, $\{z_\alpha\}$: the coordinate of $\mathfrak{p}^+$.
Integrating by parts, we can show

$$(2\pi\sqrt{-1})^{s(2s-1)}\frac{\Gamma^{\mathrm{a}}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^{\mathrm{a}}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))}\mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z} F^l_{\mathrm{a}}\left(\begin{matrix}\nu\\ \mu\end{matrix}; f; z\right)$$

$$= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \mathcal{B}^+_{\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} f(x_{\mathrm{a}})\, dx_{\mathrm{a}}\right) dy_{\mathrm{a}}$$

$$= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J z_{\mathrm{a}} J)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} \mathcal{B}^+_{-\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}} f(x_{\mathrm{a}})\, dx_{\mathrm{a}}\right) dy_{\mathrm{a}}.$$

Proof of (A2)

$$\mathcal{B}^{+}_{\lambda,\mathrm{s}} \colon \mathcal{O}(\mathfrak{p}^{+}) \longrightarrow \mathcal{O}(\mathfrak{p}^{+}) \otimes \mathfrak{p}_{\mathrm{s}}^{-},$$

$$(\mathcal{B}^{+}_{\lambda,\mathrm{s}} f)(z) := \mathrm{Proj}_{\mathfrak{p}^{-}\to\mathfrak{p}_{\mathrm{s}}^{-}}\left(\frac{1}{2}\sum_{\alpha,\beta}(e_\alpha {}^t z e_\beta + e_\beta {}^t z e_\alpha)\frac{\partial^2 f}{\partial z_\alpha \partial z_\beta}(z) + \lambda \sum_\alpha e_\alpha \frac{\partial f}{\partial z_\alpha}(z)\right),$$

where $\{e_\alpha\}$: a basis of $\mathfrak{p}^{-}$, $\{z_\alpha\}$: the coordinate of $\mathfrak{p}^{+}$.
Integrating by parts, we can show

$$(2\pi\sqrt{-1})^{s(2s-1)}\frac{\Gamma_s^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_l))}\mathcal{B}^{+}_{-\mu+2\nu+2s,\mathrm{s},z} F_{\mathrm{a}}^{l}\left(\begin{matrix}\nu \\ \mu\end{matrix}; f; z\right)$$

$$= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\mathrm{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times\left(\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}} e^{\frac{1}{2}\mathrm{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)}\mathcal{B}^{+}_{\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu} f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}$$

$$= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\mathrm{tr}(y_{\mathrm{a}} J z_{\mathrm{a}} J)}\,\mathrm{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times\left(\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^{+}} e^{\frac{1}{2}\mathrm{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}\mathcal{B}^{+}_{-\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}} f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}.$$

Proof of (A2)

$$\mathcal{B}^+_{\lambda,\mathrm{s}} \colon \mathcal{O}(\mathfrak{p}^+) \longrightarrow \mathcal{O}(\mathfrak{p}^+) \otimes \mathfrak{p}_\mathrm{s}^-,$$

$$(\mathcal{B}^+_{\lambda,\mathrm{s}} f)(z) := \operatorname{Proj}_{\mathfrak{p}^- \to \mathfrak{p}_\mathrm{s}^-} \left(\frac{1}{2} \sum_{\alpha,\beta} (e_\alpha {}^t z e_\beta + e_\beta {}^t z e_\alpha) \frac{\partial^2 f}{\partial z_\alpha \partial z_\beta}(z) + \lambda \sum_\alpha e_\alpha \frac{\partial f}{\partial z_\alpha}(z) \right),$$

where $\{e_\alpha\}$: a basis of $\mathfrak{p}^-$, $\{z_\alpha\}$: the coordinate of $\mathfrak{p}^+$.
Integrating by parts, we can show

$$\begin{aligned}
&(2\pi\sqrt{-1})^{s(2s-1)} \frac{\Gamma^\mathrm{a}_s(2\nu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma^\mathrm{a}_s(2\mu - (\underline{0}_{s-l}, \underline{1}_l))} \mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z} F^l_\mathrm{a}\left(\begin{matrix} \nu \\ \mu \end{matrix}; f; z \right) \\
&= \int_{\Omega_\mathrm{a}} e^{-\frac{1}{2}\operatorname{tr}(y_\mathrm{a} J^* z_\mathrm{a} J^*)} \operatorname{Pf}(y_\mathrm{a})^{2(\nu-\mu)} \\
&\quad \times \left(\int_{\mathfrak{a}_\mathrm{a} + \sqrt{-1}\mathfrak{n}^+_\mathrm{a}} e^{\frac{1}{2}\operatorname{tr}(x_\mathrm{a} J^* y_\mathrm{a} J^*)} \mathcal{B}^+_{\mu+2s,\mathrm{s},z_\mathrm{s}+x_\mathrm{a}} \det(z_\mathrm{s} + x_\mathrm{a})^{-\mu} f(x_\mathrm{a})\, dx_\mathrm{a} \right) dy_\mathrm{a} \\
&= \int_{\Omega_\mathrm{a}} e^{-\frac{1}{2}\operatorname{tr}(y_\mathrm{a} J z_\mathrm{a} J)} \operatorname{Pf}(y_\mathrm{a})^{2(\nu-\mu)} \\
&\quad \times \left(\int_{\mathfrak{a}_\mathrm{a} + \sqrt{-1}\mathfrak{n}^+_\mathrm{a}} e^{\frac{1}{2}\operatorname{tr}(x_\mathrm{a} J^* y_\mathrm{a} J^*)} \det(z_\mathrm{s} + x_\mathrm{a})^{-\mu} \mathcal{B}^+_{-\mu+2s,\mathrm{s},z_\mathrm{s}+x_\mathrm{a}} f(x_\mathrm{a})\, dx_\mathrm{a} \right) dy_\mathrm{a}.
\end{aligned}$$

Proof of (A2) (conti.)

$$(2\pi\sqrt{-1})^{s(2s-1)} \frac{\Gamma_s^{\mathrm{a}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\mu - (\underline{0}_{s-l}, \underline{1}_l))} \mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z} F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f; z \Big)$$

$$= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times \left(\int_{a_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^+} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} \mathcal{B}^+_{-\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}} f(x_{\mathrm{a}})\, dx_{\mathrm{a}} \right) dy_{\mathrm{a}}.$$

Since $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}_{\mathrm{a}}^+) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, by Faraut–Korányi (1990) we have

$$\left\langle f(x_{\mathrm{a}}), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_{\underline{1}_l}} f(z_{\mathrm{a}}).$$

$$\therefore \mathcal{B}^+_{\lambda,\mathrm{s}} f(z_{\mathrm{a}}) = 0 \qquad (\lambda > 4s-1 \Longrightarrow \lambda \in \mathbb{C}).$$

Therefore we get $\mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z} F_{\mathrm{a}}^l \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f; z \Big) = 0.$

Proof of (A2) (conti.)

$$(2\pi\sqrt{-1})^{s(2s-1)}\frac{\Gamma_s^{\mathrm{a}}(2\nu-(\underline{0}_{s-l},\underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\mu-(\underline{0}_{s-l},\underline{1}_l))}\mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z}F^l_{\mathrm{a}}\begin{pmatrix}\nu\\ \mu\end{pmatrix};f;z\Bigr)$$

$$=\int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}}J^*z_{\mathrm{a}}J^*)}\operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)}$$

$$\times\left(\int_{\mathfrak{a}_{\mathrm{a}}+\sqrt{-1}\mathfrak{n}_{\mathrm{a}}^+} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}}J^*y_{\mathrm{a}}J^*)}\det(z_{\mathrm{s}}+x_{\mathrm{a}})^{-\mu}\mathcal{B}^+_{-\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}}f(x_{\mathrm{a}})\,dx_{\mathrm{a}}\right)dy_{\mathrm{a}}.$$

Since $f(x_{\mathrm{a}})\in\mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}})\subset\mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, by Faraut–Korányi (1990) we have

$$\left\langle f(x_{\mathrm{a}}),e^{\operatorname{tr}(xz^*)}\right\rangle_{\lambda,x}=\frac{1}{(\lambda)_{\underline{1}_l}}f(z_{\mathrm{a}}).$$

$$\therefore\ \mathcal{B}^+_{\lambda,\mathrm{s}}f(z_{\mathrm{a}})=0\qquad(\lambda>4s-1\Longrightarrow\lambda\in\mathbb{C}).$$

Therefore we get $\mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z}F^l_{\mathrm{a}}\left(\begin{matrix}\nu\\ \mu\end{matrix};f;z\right)=0.$

Proof of (A2) (conti.)

$$
\begin{aligned}
&(2\pi\sqrt{-1})^{s(2s-1)} \frac{\Gamma_s^{\mathrm{a}}(2\nu - (\underline{0}_{s-l}, \underline{1}_l))}{\Gamma_s^{\mathrm{a}}(2\mu - (\underline{0}_{s-l}, \underline{1}_l))} \mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z} F^l_{\mathrm{a}} \begin{pmatrix} \nu \\ \mu \end{pmatrix}; f; z \Big) \\
&= \int_{\Omega_{\mathrm{a}}} e^{-\frac{1}{2}\operatorname{tr}(y_{\mathrm{a}} J^* z_{\mathrm{a}} J^*)} \operatorname{Pf}(y_{\mathrm{a}})^{2(\nu-\mu)} \\
&\quad \times \left(\int_{\mathfrak{a}_{\mathrm{a}} + \sqrt{-1}\mathfrak{n}^+_{\mathrm{a}}} e^{\frac{1}{2}\operatorname{tr}(x_{\mathrm{a}} J^* y_{\mathrm{a}} J^*)} \det(z_{\mathrm{s}} + x_{\mathrm{a}})^{-\mu} \mathcal{B}^+_{-\mu+2s,\mathrm{s},z_{\mathrm{s}}+x_{\mathrm{a}}} f(x_{\mathrm{a}})\, dx_{\mathrm{a}} \right) dy_{\mathrm{a}}.
\end{aligned}
$$

Since $f(x_{\mathrm{a}}) \in \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+_{\mathrm{a}}) \subset \mathcal{P}_{\underline{1}_l}(\mathfrak{p}^+)$, by Faraut–Korányi (1990) we have

$$
\left\langle f(x_{\mathrm{a}}), e^{\operatorname{tr}(xz^*)} \right\rangle_{\lambda,x} = \frac{1}{(\lambda)_{\underline{1}_l}} f(z_{\mathrm{a}}).
$$

$$
\therefore\ \mathcal{B}^+_{\lambda,\mathrm{s}} f(z_{\mathrm{a}}) = 0 \qquad (\lambda > 4s-1 \Longrightarrow \lambda \in \mathbb{C}).
$$

Therefore we get $\mathcal{B}^+_{-\mu+2\nu+2s,\mathrm{s},z} F^l_{\mathrm{a}} \left(\begin{matrix} \nu \\ \mu \end{matrix}; f; z \right) = 0$.