Symmetry breaking operators and orthogonal polynomials

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Branching law: discrete case

$$\pi|_{G'} \simeq \sum_{\lambda \in \hat{G}'}^{\oplus} m_\lambda \rho_\lambda \text{ with } m_\lambda \in \mathbb{N} \cup \{\infty\}$$

The coefficient m_λ is the *multiplicity* of ρ_λ in $\pi|_{G'}$, and is equal to the dimension of the space $Hom_{G'}(\pi|_{G'}, \rho_\lambda)$.

In this context, a non zero element $T_\lambda \in Hom_{G'}(\pi|_{G'}, \rho_\lambda)$ is called *symmetry breaking operator*, and the collection $(T_\lambda)_{\lambda \in \hat{G}'}$ is called *symmetry breaking transform*.

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$\rightarrow$ Spectral analysis.

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$\rightarrow$ Spectral synthesis.

Let $\lambda \in \mathbb{N}\backslash\{0;1\}$, and define the weighted Bergman space $H^2_\lambda(\Pi)$ on the Poincaré upper half-plane Π:

$$H^2_\lambda(\Pi) = \mathcal{O}(\Pi) \cap L^2(\Pi, y^{\lambda-2}\ dxdy)$$

We then define the holomorphic discrete series representation π_λ of $SL_2(\mathbb{R})$ for $f \in H^2_\lambda(\Pi)$ by the formula:

$$\pi_\lambda(g)f(z) = (cz+d)^{-\lambda} f\left(\frac{az+b}{cz+d}\right)$$

for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Geometric setting

We consider the case $(G, G') = (SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), \Delta SL_2(\mathbb{R}))$.
We are interested in the tensor product representation of two discrete series $\pi_{\lambda'} \otimes \pi_{\lambda''}$ which is considered as a representation of $SL_2(\mathbb{R})$ on the space $H^2_{\lambda'}(\Pi) \hat{\otimes} H^2_{\lambda''}(\Pi)$.

Branching law (Repka, Molchanov)

$$\pi_{\lambda'} \otimes \pi_{\lambda''}\big|_{SL_2(\mathbb{R})} \simeq \sum_{l \in \mathbb{N}}^{\oplus} \pi_{\lambda'+\lambda''+2l}$$

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Multiplicity free branching law

$$\Leftrightarrow$$

dim $Hom_{SL_2(\mathbb{R})}(\pi_{\lambda'} \otimes \pi_{\lambda''}|_{SL_2(\mathbb{R})}, \pi_{\lambda'''}) \leq 1$ for all $\lambda',\ \lambda'',\ \lambda'''$.

Let $\lambda',\ \lambda'',\ \lambda''' \in \mathbb{N}\backslash\{0;1\}$ such that $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$.
For $f \in H^2_{\lambda'}(\Pi)\hat{\otimes} H^2_{\lambda''}(\Pi)$, we define the Rankin-Cohen brackets as :

Rankin-Cohen bi-differential operators

$$RC^{\lambda'''}_{\lambda',\lambda''}(f)(z) = \sum_{j=0}^{l} \frac{(-1)^j(\lambda' + l - j)_j(\lambda'' + j)_{l-j}}{j!(l-j)!} \frac{\partial^l f}{\partial z_1^{l-j}\partial z_2^j}\bigg|_{z_1=z_2=z}$$

Each operator $RC^{\lambda'''}_{\lambda',\lambda''}$ is a *symmetry breaking operator* from $H^2_{\lambda'}(\Pi)\hat{\otimes} H^2_{\lambda''}(\Pi)$ to $H^2_{\lambda'''}(\Pi)$ and it is related to Jacobi polynomials through the F-method [1].

[1] T. Kobayashi and M. Pevzner. Differential symmetry breaking operators: II. Rankin-Cohen operators for symmetric pairs. *Selecta Math. (N.S.)*, 22(2):847-911, 2016.

We define the relative reproducing kernel $K^{\lambda'''}_{\lambda',\lambda''}$, for $w_1,\ w_2,\ z \in \Pi$:

$$K^{\lambda'''}_{\lambda',\lambda''}(z, w_1, w_2) = (w_2 - w_1)^l \left(\frac{w_1 - \bar{z}}{2i}\right)^{-(\lambda'+l)} \left(\frac{w_2 - \bar{z}}{2i}\right)^{-(\lambda''+l)}$$

Theorem (L.)

Set $\lambda',\ \lambda'',\ \lambda''' \in \mathbb{N}\backslash\{0;1\}$ *such that* $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. *Let* $w_1,\ w_2 \in \Pi$, *and* $g \in H^2_{\lambda'''}(\Pi)$. *Then the adjoint operator* $\left(RC^{\lambda'''}_{\lambda',\lambda''}\right)^*$ *is given by :*

$$(RC^{\lambda'''}_{\lambda',\lambda''})^* g(w_1, w_2) = C \int_{\Pi} g(z) K^{\lambda'''}_{\lambda',\lambda''}(z, w_1, w_2) d\mu(z)$$

And this is a holographic operator.

Set $\lambda', \lambda'', \lambda''' \in \mathbb{N}\backslash\{0;1\}$ such that $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Define the operator $\Psi^{\lambda'''}_{\lambda',\lambda''}$ on the space $H_{\lambda'''}(\Pi)$ by

$$\Psi^{\lambda'''}_{\lambda',\lambda''}(g)(w_1, w_2) = \frac{(w_1 - w_2)^l}{2^{\lambda'+\lambda''+2l-1} l!} \int_{-1}^{1} g\left(\frac{w_1 - w_2 + t(w_1 + w_2)}{2}\right)(1-v)^{\lambda'+l-1}(1+v)^{\lambda''+l-1} dv$$

Theorem (T.Kobayashi,M.Pevzner[2])

The operator $\Psi^{\lambda'''}_{\lambda',\lambda''}$ is a holographic operator from $H_{\lambda'''}(\Pi)$ to $H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi)$.

[2] T. Kobayashi and M. Pevzner. Inversion of Rankin-Cohen operators via holographic transform. *Annales de l'Institut Fourier*, 2020.

For $\lambda \in \mathbb{N}\backslash\{0;1\}$, we consider the space $L^2(\mathbb{R}^+, t^{\lambda-1}\ dt) := L^2_\lambda(\mathbb{R}^+)$. Then the Laplace transform defined, for $f \in L^2_\lambda(\mathbb{R}^+)$, by :

$$\mathcal{F}f(z) = \int_{\mathbb{R}^+} f(t)e^{itz}t^{\lambda-1}\ dt$$

is an one-to-one isometry (up to a scalar) from $L^2_\lambda(\mathbb{R}^+)$ to $H^2_\lambda(\Pi)$. This allows us to realize the holomorphic discrete series representations of $SL_2(\mathbb{R})$ on this space.

L^2-model

The following diagram clarifies the situation:

$$
\begin{tikzcd}
L^2_{\lambda'}(\mathbb{R}^+)\hat{\otimes}L^2_{\lambda''}(\mathbb{R}^+) \arrow[r, "\widehat{RC^{\lambda'''}_{\lambda',\lambda''}}"] \arrow[d, "\mathcal{F}\otimes\mathcal{F}"'] & L^2_{\lambda'''}(\mathbb{R}^+) \arrow[d, "\mathcal{F}"] \\
H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi) \arrow[r, "RC^{\lambda'''}_{\lambda',\lambda''}"'] & H^2_{\lambda'''}(\Pi)
\end{tikzcd}
$$

A geometric transformation

A "stratification" of the cone $\mathbb{R}^+ \times \mathbb{R}^+$

We use the diffeomorphism θ from $\mathbb{R}^+ \times (-1,1)$ to $\mathbb{R}^+ \times \mathbb{R}^+$:

$$\theta(t,v) = (\frac{t}{2}(1-v), \frac{t}{2}(1+v))$$

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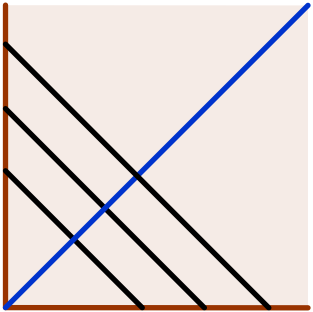
This corresponds to the following "stratification" of the cone $\mathbb{R}^+ \times \mathbb{R}^+$:



A geometric transformation

Using this diffeomorphism, one can show the following isomorphism

$$\begin{aligned} & L^2_{\lambda'}(\mathbb{R}^+)\hat{\otimes}L^2_{\lambda''}(\mathbb{R}^+) \\ \simeq \quad & L^2\left(\mathbb{R}^+, t^{\lambda'+\lambda''-1}\ dt\right)\hat{\otimes}L^2\left((-1;1),(1-v)^{\lambda'-1}(1+v)^{\lambda''-1}\ dv\right) \end{aligned}$$

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\simeq\ & \sum_{l\geq 0}^{\oplus} L^2\left(\mathbb{R}^+, t^{\lambda'+\lambda''-1}\ dt\right)\hat{\otimes}\ \mathbb{C}\cdot P_l^{\lambda'-1,\lambda''-1}(v)
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\end{aligned}$$

Theorem (L.)

The orthogonal projection on the space $L^2\left(\mathbb{R}^+, t^{\lambda'+\lambda''-1}\, dt\right)\hat{\otimes}\ \mathbb{C}\cdot P_l^{\lambda'-1,\lambda''-1}(v)$ *is a symmetry breaking operator for the* $SL_2(\mathbb{R})$ *representation realized on* $L^2\left(\mathbb{R}^+, t^{\lambda'+\lambda''-1}\, dt\right)\hat{\otimes} L^2\left((-1;1), (1-v)^{\lambda'-1}(1+v)^{\lambda''-1}\, dv\right)$.

Remarks

The orthogonal projection is actually the classical Jacobi transform (up to a constant):

$$\mathcal{J}_l^{(\lambda',\lambda'')} f(t,v) = P_l^{\lambda'-1,\lambda''-1}(v) \int_{-1}^{1} f(t,u) P_l^{\lambda'-1,\lambda''-1}(u)\, d\mu_{\lambda',\lambda''}(u).$$

So we can restate the theorem as:

Corollary

The Jacobi transform $\mathcal{J}_l^{(\lambda',\lambda'')}$ is a symmetry breaking transform associated to the tensor product of two holomorphic discrete series representations for the pair $(SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), \Delta SL_2(\mathbb{R}))$.

Remarks

The previous theorem is equivalent to the following formula about special functions:

Corollary/Lemma

For $x > 0$ and $u \in (-1, 1)$:

$$C x^l \ {}_0F_1(\lambda' + \lambda'' + 2l; x) P_l^{\lambda'-1, \lambda''-1}(u) =$$
$$\int_{-1}^{1} {}_0F_1\left(\lambda'; \frac{x}{4}(1-u)(1-v)\right) {}_0F_1\left(\lambda''; \frac{x}{4}(1+u)(1+v)\right) P_l^{\lambda'-1, \lambda''-1}(v) \, d\mu_{\lambda', \lambda''}(v),$$

where $C = 2^{\lambda'+\lambda''-1} \frac{\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\lambda''')}$ and $d\mu_{\lambda', \lambda''}(v) = (1-v)^{\lambda'-1}(1+v)^{\lambda''-1} \, dv$.

This formula allows to find back a formula of Bateman which describe the product of two ${}_0F_1$ hypergeometric functions.

n-fold tensor product of holomorphic discrete series representations

Let $\Lambda = (\lambda_1, \cdots, \lambda_n)$ such that $\lambda_i \in \mathbb{N}\backslash\{0;1\}$ and $|\Lambda| = \sum \lambda_i$. Then we have the following branching law:

Branching law

$$\bigotimes_{i=1}^{n} \pi_{\lambda_i} \simeq \sum_{k\in\mathbb{N}}^{\oplus} \binom{n+k-2}{n-2} \pi_{|\lambda|+2k}$$

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This is not multiplicity free anymore so we need to build a basis of $Hom_{SL_2(\mathbb{R})}(\bigotimes_{i=1}^{n} \pi_{\lambda_i}, \pi_{|\Lambda|+2k})$ for all $k \in \mathbb{N}$.

We start from the L^2-model for the representation $\bigotimes_{i=1}^{n} \pi_{\lambda_i}$, so the representation acts on the space $L^2((\mathbb{R}^+)^n, \prod_{i=1}^{n} t_i^{\lambda_i - 1}\, dt_i)$, and we are going to introduce a "stratified" model for this tensor product representation.

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For this, we introduce a $(n-1)$-dimensional simplex:

$$D_{n-1} = \{(v_1, \cdots, v_{n-1}) \mid v_i > 0 \text{ and } 1 - |v| > 0\},$$

where $|v| = \sum_{i=1}^{n-1} v_i$, and we define the diffeomorphism ι from $\mathbb{R}^+ \times D_{n-1}$ to $(\mathbb{R}^+)^n$:

$$\iota(t, (v_1, \cdots, v_{n-1})) = (tv_1, \cdots, tv_{n-1}, -t|v|)$$

We use ι to get the following isomorphism of Hilbert spaces

$$L^2((\mathbb{R}^+)^n, \prod_{i=1}^{n} t_i^{\lambda_i - 1}\ dt_i) \simeq L^2(\mathbb{R}^+, t^{|\Lambda|-1} dt) \hat{\otimes} L^2(D_{n-1}, (1-|v|)^{\lambda_n - 1} \prod_{i=1}^{n-1} v_i^{\lambda_i - 1}\ dv)$$

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The space $L^2(D_{n-1}, (1-|v|)^{\lambda_n - 1}\prod_{i=1}^{n-1} v_i^{\lambda_i - 1}\ dv)$ admits a Hilbert basis of orthogonal polynomials, and we denote by $Pol_k(D_{n-1})$ the space of polynomials of degree k which are orthogonal to all the polynomials of lesser degree.

Theorem (L.)

Let $k \in \mathbb{N}$. *The space* $Hom_{SL_2(\mathbb{R})}(\bigotimes_{i=1}^{n} \pi_{\lambda_i}, \pi_{|\Lambda|+2k})$ *is isomorphic to* $Pol_k(D_{n-1})$.
More precisely, for a polynomial P *define the operator* $\Psi_k^\Lambda(P)$ *for* $f \in L^2(\mathbb{R}^+, t^{|\Lambda|-1}dt)\hat{\otimes} L^2(D_{n-1}, (1-|v|)^{\lambda_n - 1} \prod_{i=1}^{n-1} v_i^{\lambda_i - 1}\, dv)$ *by:*

$$\Psi_k^\Lambda(P)f(t) = t^{-k} \int_{D_{n-1}} f(t,v)P(v)(1-|v|)^{\lambda_n - 1} \prod_{i=1}^{n-1} v_i^{\lambda_i - 1}\, dv$$

.
Then $\Psi_k^\Lambda(P) \in Hom_{SL_2(\mathbb{R})}(\bigotimes_{i=1}^{n} \pi_{\lambda_i}, \pi_{|\Lambda|+2k})$ *iff* $P \in Pol_k(D_{n-1})$.

In order to prove this theorem, we used the following family of polynomials

Orthogonal polynomials on the simplex

Let $\underline{k} = (k_1, \cdots, k_{n-1})$ such that $|\underline{k}| = k$.

$$P_{\underline{k}}^{\Lambda}(v_1, \cdots, v_{n-1}) = P_{k_{n-1}}^{\lambda_n - 1, \alpha_{n-1} - 1}(2|v| - 1) \prod_{i=1}^{n-1} (|v^{(i)}| + v_{i+1})^{k_i} P_{k_i}^{\lambda_{i+1} - 1, \alpha_i - 1}\left(\frac{|v^{(i)}| - v_{i+1}}{|v^{(i)}| + v_{i+1}}\right)$$

where $\alpha_i = |\Lambda^{(i)}| + 2|\underline{k}^{(i-1)}|$ and the notation $v^{(i)} = (v_1, \cdots, v_i)$.

Conclusion

1. Similar result can be obtained for the pair $(SO(2,n), SO(2,n-1)$ with $n \geq 3$ and, by induction, for the pair $(SO(2,n), SO(2,n-k))$. In this case, we meet the family of Gegenbauer polynomials.

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2. All these examples fit in a more general setting using the language of Jordan algebras, symmetric cones and tube domains. It allows to work on the restriction of representation of the holomorphic discrete series of scalar type for Lie groups of tube type.

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2. All these examples fit in a more general setting using the language of Jordan algebras, symmetric cones and tube domains. It allows to work on the restriction of representation of the holomorphic discrete series of scalar type for Lie groups of tube type.
3. Orthogonal polynomials seems to be useful to understand abstract branching laws in this context.

Thank you for your attention.