Differential symmetry breaking operators for $(O(n+1,1), O(n,1))$ for differential forms

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Representation Theory and Noncommutative Geometry
AIM Research Community

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Based on a joint work with Toshiyuki Kobayashi and Michael Pevzner
([Kobayashi–Kubo–Pevzner, Lecture Notes in Math., 2016])

Differential symmetry breaking operators

differential symmetry breaking operators (DSBOs)

- Symmetry breaking operators:

$$\begin{array}{lcl} G \curvearrowright X & \Longrightarrow & G \curvearrowright C^\infty(X, \mathcal{V}) \\ \cup & & \cup \circlearrowright \qquad \downarrow T \\ G' \curvearrowright Y & \Longrightarrow & G' \curvearrowright C^\infty(Y, \mathcal{W}) \end{array}$$

Definition (T. Kobayashi)

differential symmetry breaking operators
= G'-intertwining differential operators $T\colon C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$

Goal

Classify DSBOs with their explicit formulas in a specific setting.

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2. Main Results
3. F-method (algebraic Fourier transform of Verma modules)

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$C^\infty(X, \mathcal{V}) := \mathcal{E}^i(S^n)$ (the space of i-forms on S^n)

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where Ω : a positive-valued function in $C^\infty(G \times S^n)$ (*conformal factor*)

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Conformal representation $\varpi^{(i)}_{u,\delta}$ of $G = \mathrm{Conf}(S^n)$

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  - $u \in \mathbb{C}$, $\delta \in \mathbb{Z}/2\mathbb{Z}$,
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Note:

$(\varpi^{(i)}_{u,\delta}, \mathcal{E}^i(S^n))$: a p.s. representation of $O(n+1,1)$

$(\varpi^{(j)}_{v,\varepsilon}, \mathcal{E}^j(S^{n-1}))$: a p.s. representation of $O(n,1)$

$$\mathcal{E}^i(S^n)_{u,\delta} \equiv (\varpi^{(i)}_{u,\delta}, \mathcal{E}^i(S^n)), \quad \mathcal{E}^j(S^{n-1})_{v,\varepsilon} \equiv (\varpi^{(j)}_{v,\varepsilon}, \mathcal{E}^j(S^{n-1}))$$

$\mathrm{Diff}_{G'}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}) :=$ the space of differential SBOs

Main Problems:

(1) Classify $(i, j, u, v, \delta, \varepsilon)$ so that

$$\mathrm{Diff}_{G'}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}) \neq \{0\}.$$

(2) Determine

$$\dim_{\mathbb{C}} \mathrm{Diff}_{G'}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}).$$

(3) Construct

$$D \in \mathrm{Diff}_{G'}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon})$$

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$\mathrm{Diff}_{G'}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon})$:= the space of differential SBOs

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2. Main Results I: Classification (Problems (1) and (2))

Theorem (Kobayashi–K–Pevzner, [Lecture Notes in Math., 2016])

For $(i, j, u, v, \delta, \varepsilon)$ with $n \geq 3$, TFAE:

(i) $\mathrm{Diff}_{G'}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}) \neq \{0\}$;

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Case $\underline{j = i - 1}$:

- $1 \leq i \leq n$;
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Notes: The Hodge star operator reduces all the cases to $j = i-1, i+1$.

Main Results II: Explicit Formula (Problem (3))

Representation theory:

K-picture $\Longrightarrow$ N-picture

Conformal geometry:

Understand S^n as the conformal compactification of the flat Riemannian manifold $\mathbb{R}^n$.

$$\begin{array}{ccccc}
\mathcal{E}^i(S^n)_{u,\delta} & \longrightarrow & \mathcal{E}^j(S^{n-1})_{v,\varepsilon} & \qquad & S^n \\
\big\downarrow & & \big\downarrow & & \big\uparrow \text{dense} \\
\mathcal{E}^i(\mathbb{R}^n) & \xrightarrow[\mathcal{D}^{i\to j}]{\cdots\cdots} & \mathcal{E}^j(\mathbb{R}^{n-1}) & & \mathbb{R}^n
\end{array}$$

Cases: $j = i - 1$

For $u \in \mathbb{C}$ and $a \in \mathbb{N}_+$, we set

$$\mathcal{D}^{i \to i-1}_{u,a} := \mathrm{Rest}_{x_n=0} \circ \left(\mathcal{D}^{\mu+1}_{a-2} dd^* \iota_{\frac{\partial}{\partial x_n}} + p\mathcal{D}^{\mu+1}_{a-1} d^* - q\mathcal{D}^{\mu}_{a} \iota_{\frac{\partial}{\partial x_n}} \right)$$

with

- p, q: appropriate constants, $\mu := u + i - \frac{n-1}{2}$,
- $\mathcal{D}^{\mu+1}_{a-2}$, $\mathcal{D}^{\mu+1}_{a-1}$, $\mathcal{D}^{\mu}_{a}$: scalar-valued differential operators of order $a-2$, $a-1$, a, respectively.

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On the scalar-valued differential operators $\mathcal{D}_a^{\mu}$:

1. These are polynomials of
   - $\Delta_{\mathbb{R}^{n-1}}$ (the Laplacian for the hyperplane $\mathbb{R}^{n-1}$),
   - $\frac{\partial}{\partial x_n}$ (the normal derivative with respect to the hyperplane $\mathbb{R}^{n-1}$)

   with degree a.

2. The coefficients of them coincide with those for Gegenbauer polynomials.

$\widetilde{\mathcal{D}}_{u,a}^{i\to i-1}$:= renormalized operator such that $\widetilde{\mathcal{D}}_{u,a}^{i\to i-1} \neq 0$ for any (i, a, u).

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- $a = 1 - u$ with $u = 0 \quad (1 \leq i \leq n-2)$
- $a = 1 - u$ with $u \in -\mathbb{N}$ $(i = 0)$

Theorem (Kobayashi–K–Pevzner, [Lecture Notes in Math., 2016])

The following hold:

1. The differential operator $\widetilde{\mathcal{D}}_{u,a}^{i\to i\pm1}$ can be extended to S^n and the extended differential operator is an SBO.
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Brief outline of the proofs of the main results

1. Reformulate the main problems (conformal geometry) to ones in representation theory.

2. Use the F-method (cf. [Kobayashi '13], [Kobayashi–Pevzner '16])

    - The "F" for the F-method stands for the "*Fourier transform*".

      (The algebraic Fourier transform of the Verma modules)

    - We enhanced the F-method to the vector-valued case.

3. Interpret the results back in conformal geometry.

Remark:

Fischmann–Juhl–Somberg also independently classified the DSBOs for differential forms for $(SO_0(n+1,1), SO_0(n,1))$ (AMS Memoirs, 2020).

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3. F-method (cf. Kobayashi–Pevzner, 2016)

Naive idea for the F-method

The F-method is a technique to classify and also construct

$$D \in \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$$

by solving a system of PDEs via the algebraic Fourier transform.

Theorem (Kobayashi–Pevzner '16)

There exist natural linear isomorphisms

$$\mathrm{Sol(PDE)} \xleftarrow{\sim} \mathrm{Hom(Verma\ modules)} \xrightarrow{\sim} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

- Sol(PDE): the space of poly. solutions to a system of certain PDEs

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Duality theorem (Hom $\xrightarrow{\sim}$ Diff)

$$\begin{array}{ccccc} G & \supset & H & \Longrightarrow & G/H := X \\ \cup & & \cup & & \\ G' & \supset & H' & \Longrightarrow & G'/H' := Y \end{array}$$

- V, W: f.d. representations of H and H', resp.
- $\mathcal{V}_X := G \times_H V \to X$, $\qquad \mathcal{W}_Y := G' \times_{H'} W \to Y$,

$\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V^\vee$, $\quad \mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee) := U(\mathfrak{g}') \otimes_{U(\mathfrak{h}')} W^\vee$

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Remarks:

- The duality theorem is known when $G' = G$ and $H' = H = \text{Borel}$, especially in the setting of complex flag variaties.

  cf.
  - Kostant ('75)
  - Harris–Jakobsen ('82)

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Steps for the F-method (Sol $\stackrel{\sim}{\leftarrow}$ Hom)

Preparation:

(1) Algebraic Fourier transform $\widehat{\cdot}$ of the Weyl algebra

(2) Lie algebra homomorphism $\widehat{d\pi_\mu}$

(3) Algebraic Fourier transform F_c of Verma modules

Then

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(1) Algebraic Fourier transform of the Weyl algebra

- V: f.d. vector space over $\mathbb{C}$ with $\dim_{\mathbb{C}} V = n$
- $(z_1, \ldots, z_n)$: coordinate for V
- $\mathcal{D}(V) := \mathbb{C}[z_1, \ldots, z_n, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}]$: the Weyl algebra of V

Definition

The *algebraic Fourier transform of $\mathcal{D}(V)$* is an algebra isomorphism

$$\mathcal{D}(V) \xrightarrow{\sim} \mathcal{D}(V^\vee), \quad T \mapsto \widehat{T}$$

induced by

$$\widehat{\frac{\partial}{\partial z_\ell}} := -\zeta_\ell, \quad \hat{z}_\ell := \frac{\partial}{\partial \zeta_\ell}, \quad 1 \leq \ell \leq n.$$

Ex: $E_z = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \Longrightarrow \widehat{E_z} = -E_\zeta - n$

Summary

(2) Lie algebra homomorphism $\widehat{d\pi_\mu}$

- G : real reductive Lie group
  $\cup$ $\implies G/P =: X$
  $P = MAN_+$: parabolic subgroup of G
- $\mathfrak{g}(\mathbb{R}) = \mathfrak{n}_-(\mathbb{R}) + \mathfrak{m}(\mathbb{R}) + \mathfrak{a}(\mathbb{R}) + \mathfrak{n}_+(\mathbb{R})$
- $\mathfrak{g} := \mathfrak{g}(\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$
- (λ, V): f.d. representation of P
- $\mathcal{V}_X := G \times_P V \to X$
- $\pi_\lambda := \mathrm{Ind}_P^G(\lambda) \curvearrowright C^\infty(X, \mathcal{V}_X)$
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- $\mathfrak{n}_-(\mathbb{R}) \simeq N_- \hookrightarrow G/P = X$
- $C^\infty(X, \mathcal{V}_X) \longrightarrow C^\infty(\mathfrak{n}_-(\mathbb{R})) \otimes V$
- $\mathfrak{g} \overset{d\pi_\lambda}{\curvearrowright} C^\infty(\mathfrak{n}_-(\mathbb{R})) \otimes V$

$$d\pi_\lambda : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V)$$

$$\widehat{d\pi_\lambda} : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V)$$

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(3) Algebraic Fourier transform F_c of Verma modules

- $\widehat{d\pi_\mu}(C) \curvearrowright \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee, \quad C \in \mathfrak{g}$
- $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee$

Theorem (Kobayashi–Pevzner '16)

There exists a $(\mathfrak{g}, P)$-module isomorphism:

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Definition

We call F_c the *algebraic Fourier transform of Verma modules*.

Recall that

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Theorem (Kobayashi–Pevzner '16)

There exists a natural isomorphism

$$F_c \otimes \mathrm{id}_W : \mathrm{Hom}_{\mathfrak{g}', P'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \mathrm{Sol}(\mathfrak{n}_+; V, W).$$

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Summary

In summary we have the following:

$$\begin{array}{ccc} & \mathrm{Sol}(\mathfrak{n}_+; V, W) & \\ {\scriptstyle F_c \otimes \mathrm{id}} \nearrow \sim & & \sim \searrow \\ \mathrm{Hom}_{\mathfrak{g}', P'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & \xrightarrow[D_{X \to Y}]{\sim} & \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y). \end{array}$$

Note: It is not necessary for $\mathfrak{n}_+$ to be abelian.

Summary

In summary we have the following:

$$\mathrm{Hom}_{\mathfrak{g}',P'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \xrightarrow[\sim]{F_c \otimes \mathrm{id}} \mathrm{Sol}(\mathfrak{n}_+; V, W) \overset{\sim}{\dashrightarrow} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$$

$$\mathrm{Hom}_{\mathfrak{g}',P'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \underset{D_{X\to Y}}{\overset{\sim}{\longrightarrow}} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

Note: It is not necessary for $\mathfrak{n}_+$ to be abelian.

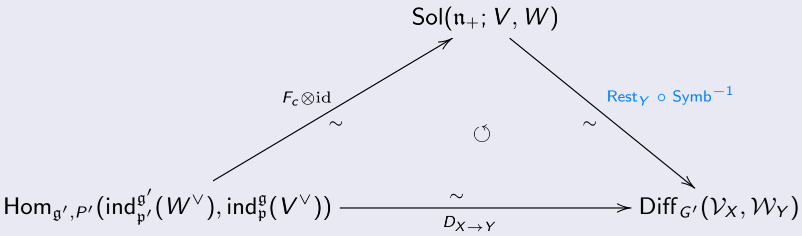
The case: $\mathfrak{n}_+$ is abelian.

Theorem (Kobayashi–Pevzner '16)

Suppose that $\mathfrak{n}_+$ is abelian. Then there exists a natural isomorphism

$$\mathrm{Rest}_Y \circ \mathrm{Symb}^{-1} \colon \mathrm{Sol}(\mathfrak{n}_+; V, W) \xrightarrow{\sim} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

Moreover the following diagram commutes:



Formula AFTW

References

- M. Fischmann, A. Juhl, and P. Somberg, *Conformal Symmetry Breaking Differential Operators on Differential Forms*, Mem. Amer. Math. Soc., vol. **268**, 2020.
- T. Kobayashi, T. Kubo, and M. Pevzner, *Conformal Symmetry Breaking Operators for Differential Forms on Spheres*, Lecture Notes in Math., vol. **2170**, Springer, 2016.
- T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators. I. General theory and F-method; II. Rankin–Cohen operators for symmetric pairs*, Selecta Math. (N.S.), **22**, (2016), pp. 801–845, 847–911.
- T. Kobayashi and B. Speh, *Symmetry Breaking for Representations of Rank One Orthogonal Groups*. Mem. Amer. Math. Soc., vol. **238**, 2015.
- T. Kobayashi and B. Speh, *Symmetry Breaking for Representations of Rank One Orthogonal Groups II*, Lecture Notes in Math., vol. **2234**, Springer, 2018.

Thank you for your attention!