Spectral theory of automorphic forms

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Connections with

- Representation theory of reductive groups over local and global fields.
- Partial differential equations on locally symmetric spaces.
- Number theory.

1. Automorphic forms, automorphic representations

Basic set up

- G connected semisimple Lie group, non-compact type.
- H Lie group, $\Pi(H)$ equivalence classes of irreducible unitary representations of H.
- $K \subset G$ maximal compact subgroup,
- $\widetilde{X} = G/K$ Riemannian symmetric space.
- $\Gamma \subset G$ lattice, i.e., discrete subgroup with $\mathrm{vol}(\Gamma\backslash G) < \infty$.
- $X = \Gamma\backslash\widetilde{X}$ locally symmetric space, manifold if Γ is torsion free.
- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$.
- $\Omega \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ Casimir element.

Automorphic forms

- $\phi \in C^\infty(\Gamma\backslash G)$ is an automorphic form, if ϕ is K-finite, $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$-finite, and of moderate growth.

Example. $\phi \in L^2(\Gamma\backslash G)^K \cong L^2(\Gamma\backslash\widetilde{X})$, joint eigenfunction of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$.

Cusp forms

- $P \subset G$ parabolic subgroup, $P = M_P A_P N_P$, M_P reductive, $A_P \cong (\mathbb{R}^+)^k$, N_P nilpotent.
- P is called cuspidal parabolic, if $(\Gamma \cap N_P)\backslash N_P$ is compact.
- $\phi \in C^\infty(\Gamma\backslash G)$ automorhic form, ϕ is called cusp form, if

$$\int_{(\Gamma\cap N_P)\backslash N_P} f(ng)dn = 0$$

for all proper cuspidal parabolic subgroups P of G.

Representation theory

- R_Γ right regular representation of G in $L^2(\Gamma\backslash G)$:

$$(R_\Gamma(g)f)(g') = f(g'g), \quad f \in L^2(\Gamma\backslash G).$$

- Theory of Eisenstein series implies decomposition in invariant subspaces

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{dis}}(\Gamma\backslash G) \oplus L^2_{ac}(\Gamma\backslash G)$$

- $L^2_{\mathrm{dis}}(\Gamma\backslash G)$ maximal invariant subspace, spanned by irreducible subrepresentations,

$$R_{\Gamma,\mathrm{dis}} \cong \widehat{\bigoplus}_{\pi\in\Pi(G)} m_\Gamma(\pi)\pi.$$

- $m_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, R_\Gamma) = \dim \mathrm{Hom}_G(\pi, R_{\Gamma,\mathrm{dis}})$.
- $m_\Gamma(\pi) < \infty$ for all $\pi \in \Pi(G)$.

Main problem: Study of the multiplicities $m_\Gamma(\pi)$.

- Apart from special cases, one cannot hope to describe $m_\Gamma(\pi)$ explicitly.
- There exist formulas for $m_\Gamma(\pi)$ if π is a discrete series representation.

Further problems

- Generalized Ramanujan conjecture.
- Residual spectrum: $L^2_{\text{cus}}(\Gamma\backslash G) \subset L^2_{\text{dis}}(\Gamma\backslash G)$ subspace of cusps forms.
$$L^2_{\text{dis}}(\Gamma\backslash G) = L^2_{\text{cus}}(\Gamma\backslash G) \oplus L^2_{\text{res}}(\Gamma\backslash G)$$
- $L^2_{\text{res}}(\Gamma\backslash G)$ residual subspace.
- Mœglin, Waldspurger: Determination of the residual spectrum for $G = \mathrm{GL}(n)$.
- What can be said about the residual spectrum for general G?

2. Asymptotic behavior of automorphic spectra

- Study behavior of multiplicities with respect to the growth of various parameters such as the infinitesimal character or/and the level of congruence subgroups.

Examples:

a) Weyl law.

- For $\sigma \in \Pi(K)$ let
$$\Pi(G;\sigma) = \left\{\pi \in \Pi(G) \colon [\pi|_K : \sigma] > 0\right\}.$$
- $\lambda_\pi = \pi(\Omega)$ Casimir eigenvalue of $\pi \in \Pi(G)$.
$$N_\Gamma(\lambda;\sigma) = \sum_{\substack{\pi\in\Pi(G;\sigma)\\ |\lambda_\pi|\le\lambda}} m_\Gamma(\pi).$$

Problem: Behavior of $N_\Gamma(\lambda;\sigma)$ as $\lambda \to \infty$.

Equivalent problem:

- $\sigma = 1$, $\widetilde{X} = G/K$, $X = \Gamma\backslash\widetilde{X}$, $L^2(\Gamma\backslash G)^K \cong L^2(\Gamma\backslash G/K) = L^2(X)$, $\Delta \colon C^\infty(X) \to C^\infty(X)$ Laplace operator.

Lemma (Kuga): $-R(\Omega) = \Delta$.

- $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues of Δ in $L^2(X)$.
- $N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda\}$

Problem: Study behavior of $N_\Gamma(\lambda)$ as $\lambda \to \infty$.

b) Limit multiplicity problem.

- Let $\Pi(G)$ be equipped with the Fell topology.

Define a measure on $\Pi(G)$ by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi \in \Pi(G)} m_\Gamma(\pi)\delta_\pi,$$

where δ_π is the delta distribution.

Problem: Study the behavior of μ_Γ as $\mathrm{vol}(\Gamma\backslash G) \to \infty$.

- Distribution of Hecke eigenvalues
- Sato-Tate conjecture for modular forms.
- Analytic torsion, Approximation of L^2-invariants
- Families of automorphic forms (Sarnak).

2. Conjectures and results

a) Wely law

i) Γ cocompact. $n = \dim X$. Then the following Weyl law holds

$$N_\Gamma(\lambda;\sigma) = \frac{\dim(\sigma)\operatorname{vol}(\Gamma\backslash G)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2} + O(\lambda^{(n-1)/2}), \quad \lambda\to\infty.$$

- $\sigma = 1$: Avakumović, Hörmander: general elliptic operators.

ii) $\Gamma\backslash G$ non-compact:

Problem: Nonempty continuous spectrum.

- Continuous spectrum is described by Eisenstein series.

Arithmetic groups.

Let $\mathbf{G} \subset \mathrm{GL}_n$ be a semisimple algebraic group over $\mathbb{Q}$. $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called arithmetic, if Γ is commensurable to $\mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

Example: Principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$:

$$\Gamma(N) := \left\{ \gamma \in \mathrm{SL}(2,\mathbb{Z}) \colon \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod (N) \right\}$$

Theorem (Margulis): Let $\operatorname{rank}_{\mathbb{R}} G > 1$ and $\Gamma \subset G$ an irreducible lattice. Then Γ is arithmetic.

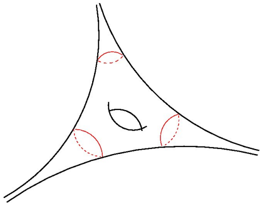
Theorem (Serre): Let $n \geq 3$. Every arithmetic subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$ contains a principal congruence subgroup.

Conjecture (Sarnak): Weyl law holds for congruence subgroups or, more generally, arithmetic groups.

Results.

a) The real rank one case. Selberg 1954,

- $X = \Gamma\backslash\mathbb{H}$, $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ lattice.
- X is a hyperbolic surface of finite area.



- $\Delta\colon C^\infty(X) \to C^\infty(X)$ Laplace operator, essentially selfadjoint in $L^2(X)$,
- Continuous spectrum of Δ: $[1/4, \infty)$.
- $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues.

- Eigenvalues $\lambda_j \geq 1/4$ embedded in the continuous spectrum, unstable w.r.t. perturbations, difficult to study.

Eisenstein series. $\Gamma = \mathrm{SL}(2, \mathbb{Z})$,

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(z))^s = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}}, \quad \mathrm{Re}(s) > 1.$$

- $E(z,s)$ admits meromorphic extension to $\mathbb{C}$, holomorphic for $\mathrm{Re}(s) = 1/2$.
- $\Delta E(z,s) = s(1-s)E(z,s)$
- $\int_0^1 E(x+iy,s)dx = y^s + c(s)y^{1-s}$
- $c(s)$ “scattering matrix”.

Heat equation method: Apply **Selberg trace formula** to $e^{-t\Delta}$. As $t \to 0$ we get

$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t}\frac{c'}{c}(1/2+ir)dr \sim \frac{\mathrm{vol}(X)}{4\pi}t^{-1}$$

Application of Karamata's Theorem gives

$$N_\Gamma(\lambda) - \frac{1}{4\pi}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{c'}{c}(1/2+ir)dr \sim \frac{\mathrm{vol}(X)}{4\pi}\lambda$$

as $\lambda \to \infty$.

- Similar for any $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$.
- $\Gamma = \mathrm{SL}(2,\mathbb{Z})$: Then the scattering matrix is given by

  $$c(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

  where $\zeta(s)$ is the Riemann zeta function.
- For $\Gamma(N)$, coefficients of the scattering matrix $C(s)$ are given by fractions of Dirichlet L-functions $L(s,\chi)$.
- Using standard estimates of $L(s,\chi)$ and $\Gamma(s)$, it follows that the growth of the winding number of $c(s)$ is of lower order.

Theorem (Selberg):

$$N_{\Gamma(N)}(\lambda) \sim \frac{\operatorname{Area}(X)}{4\pi}\lambda^2, \quad \lambda \to \infty.$$

b) Higher rank.

- Selberg trace formula is replaced by the Arthur trace formula.

The Weyl law holds in the following cases:

i) $\sigma = 1$, no estimation of the remainder term.

- S. Miller: $X = \mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.
- E. Lindenstrauss, A. Venkatesh: $\Gamma \subset G$ congruence subgroup. Use Hecke operators.

ii) $\sigma \in \Pi(K)$ arbitrary, no estimation of the remainder term.

- Mü.: $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$ principal congruence subgroup of level N, $X = \Gamma(N)\backslash \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

In progress. Let $\mathbf{G}$ be one of the following algebraic groups:

1. Inner form of $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$.
2. Symplectic, special orthogonal or unitary group.

Let $\Gamma \subset \mathbf{G}(\mathbb{R})$ be a congruence subgroup. Then the Weyl law holds for $\Gamma\backslash\mathbf{G}(\mathbb{R})/K$.

- Main ingredient of the proof: Study of the automorphic L-functions $L(\pi, s)$, $\pi \in \Pi_{\mathrm{cus}}(\mathbf{M}(\mathbb{A}))$, $\mathbf{M}$ Levi subgroup of $\mathbf{G}$, that appear in the constant terms of Eisenstein series.
- J. Arthur: Endoscopic classification of representations of classical groups.

iii) $\sigma = 1$, estimation of the remainder term.

Let $S_n = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n), \quad n \geq 2$, and let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a congruence subgroup. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Δ_Γ in $L^2(\Gamma\backslash S_n)$.

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda^2\}.$$

Thoerem 2 (Lapid, Mü.): Let $d = \dim S_n$, $N \geq 3$. Then

$$N_{\Gamma(N)}(\lambda) = \frac{\mathrm{Vol}(\Gamma(N)\backslash S_n)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^d + O(\lambda^{d-1}(\log\lambda)^{\max(n,3)})$$

as $\lambda \to \infty$.

Theorem 3 (Finis, Lapid, 2019): Let G be a simply connected, simple Chevalley group. Then there exists $\delta > 0$ such that for any congruence subgroup Γ of $G(\mathbb{Z})$ one has

$$N_{X,\mathrm{cus}}(\lambda) = \frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\lambda^d + O_\Gamma(\lambda^{d-\delta}), \quad \lambda \geq 1,$$

where $X = \Gamma\backslash G(\mathbb{R})/K$, $d = \dim(X)$.

b) Limit multiplicities

- μ_{PL} Plancherel measure on $\Pi(G)$, support of μ_{PL} is the tempered dual $\Pi_{\mathrm{temp}}(G)$.
- Up to a closed subset of Plancherel measure zero, the topological space $\Pi_{\mathrm{temp}}(G)$ is homeomorphic to a countable union of Eucledian spaces of bounded dimension.
- Under this homeomorphism, the Plancherel density is given by a continuous function.
- A relatively quasi-compact subset of $\Pi(G)$ is called bounded.
- $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots$ tower of normal subgroups of finite index, $\cap_j \Gamma_j = \{e\}$.

$$\mu_j = \frac{1}{\mathrm{vol}(\Gamma_j \backslash G)} \sum_{\pi \in \Pi(G)} m_{\Gamma_j}(\pi)\delta_\pi .$$

Conjecture: $\mu_j \to \mu_{PL}$ as $j \to \infty$.

Results:

a) $\Gamma \subset G$ uniform lattice.
De George-Wallach, Delorme: Answer affirmative.

b) $\Gamma \subset G$ non-uniform lattice.

Discrete series:

- Clozel, 1986, Speh, Rohlfs 1987: positive lower bounds

$$\limsup_{j\to\infty} \frac{m_{\Gamma_j}(\pi)}{\operatorname{vol}(\Gamma_j\backslash G)} \geq d_\pi > 0,$$

$\pi \in \Pi_d(G)$ (discrete series), d_π formal degree of π, under some restrictions.

- Savin: $\pi \in \Pi_d(G)$, $d(\pi)$ formal degree.

$$\lim_{j\to\infty} \mu_j(\{\pi\}) = d_\pi .$$

Theorem (Finis, Lapid, Mü.), 2014: Let $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma_n(N) \subset \mathrm{SL}(n, \mathbb{Z})$ the principal congruence subgroup of level N. Let $\mu_N := \mu_{\Gamma_n(N)}$.

1) For every Jordan measurable set $A \subset \Pi_{\mathrm{temp}}(G)$ we have

$$\mu_N(A) \to \mu_{PL}(A), \quad N \to \infty.$$

2) For every bounded subset $A \subset \Pi(G) \setminus \Pi_{\mathrm{temp}}(G)$ we have

$$\mu_N(A) \to 0, \quad N \to \infty.$$

Theorem (Finis, Lapid), 2018. Let F be a number field an $\mathcal{O}_F$ the ring of integers. The limit multiplicity holds for the family of all congruence subgroups of $\mathrm{SL}(n, \mathcal{O}_F)$.

- Results for quasi-split classical groups over a number field.

3. Approximation of L^2-invariants

Spectral invariants: X compact manifold. α invariant defined in terms of the spectrum of geometric differential operators such as the Laplace operator Δ_k on k-forms.

Examples:

- Betti number $b_k(X) = \dim \ker \Delta_k$,
- $\mathrm{Sign}(X) = \mathrm{Ind}(D_{\mathrm{sign}})$, $D_{\mathrm{sign}} := d + d^* \colon \Lambda^*_+(X) \to \Lambda^*_-(X)$ signature operator.
- Analytic torsion $T_X \in \mathbb{R}^+$

$$\log T_X = \frac{1}{2} \sum_{p=1}^{n} (-1)^p p \frac{d}{ds} \zeta_p(s)\Big|_{s=0}.$$

where

$$\zeta_p(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Tr}(e^{-t\Delta_p}) - b_p)) t^{s-1} dt, \quad \mathrm{Re}(s) > n/2.$$

L^2-invariants

Atiyah, 1976: L^2-index theorem, L^2-Betti numbers,
Lott, Mathai, 1992: L^2-torsion, Lück, 2004 : L^2-invariants.

- $\widetilde{X} \to X$ universal covering of X, $\Gamma := \pi_1(X)$, $d\tilde{\mu}$ Γ-invariant measure on $\widetilde{X}$.
- **Assume:** $\widetilde{X}$ is noncompact, i.e., Γ infinite.
- $\ell^2(\Gamma)$ Hilbert space completion of $\mathbb{C}[\Gamma]$.
- Left action of Γ on $\ell^2(\Gamma)$, $L(\gamma)\colon \ell^2(\Gamma) \to \ell^2(\Gamma)$, $\gamma \in \Gamma$.
- Group von Neumann algebra: $\mathcal{N}(\Gamma) := \mathcal{B}(\ell^2(\Gamma))^\Gamma$.
- von Neumann trace: $\mathrm{tr}_{\mathcal{N}(\Gamma)}\colon \mathcal{N}(\Gamma) \to \mathbb{C}$,

$$\mathrm{tr}_{\mathcal{N}(\Gamma)}(A) := \langle A(e), e\rangle_{\ell^2(\Gamma)}.$$

- $\mathcal{B}(L^2\Lambda^p(\widetilde{X}))^\Gamma$ von Neumann algebra.
- $F \subset \widetilde{X}$ fundamental domain for the action of Γ on $\widetilde{X}$.
- $L^2\Lambda^p(\widetilde{X}) \cong \ell^2(\Gamma) \otimes L^2\Lambda^p(F)$.

- $\mathcal{B}(L^2\Lambda^p(\widetilde{X}))^\Gamma \cong \mathcal{N}(\Gamma) \otimes \mathcal{B}(L^2\Lambda^p(F))$.
- $\mathrm{Tr}_\Gamma : \mathcal{N}(\Gamma) \otimes \mathcal{B}(L^2\Lambda^p(F)) \to \mathbb{C}$ constructed from $\mathrm{tr}_{\mathcal{N}(\Gamma)}$ on $\mathcal{N}(\Gamma)$ and the ordinary trace on $\mathcal{B}(L^2\Lambda^p(F))$, Tr_Γ possibly infinite.
- $H \subset L^2\Lambda^p(\widetilde{X})$ Γ-invariant subspace, $P \in \mathcal{B}(L^2\Lambda^p(\widetilde{X}))^\Gamma$ orthogonal projection onto H. Define
$$\dim_\Gamma H := \mathrm{Tr}_\Gamma(P).$$
- Replace dim by $\dim_\Gamma$ and Tr by Tr_Γ.

Assume that $\Gamma = \pi_1(X)$ is residual finite: There exists a tower

$$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_N \supset \cdots$$

of normal subgroups Γ_k of Γ of finite index with $\cap_{k=1}^\infty \Gamma_k = \{1\}$.

- Put $X_k := \Gamma_k \backslash \widetilde{X}$, $k \in \mathbb{N}$. $X_k \to X$ finite normal covering.
- Let α be a spectral invariant.

Problem: Does $\frac{\alpha(X_k)}{[\Gamma : \Gamma_k]}$ converge as $k \to \infty$ and if so, what is the limit?

- A natural candidate for the limit is the L^2-invariant $\alpha^{(2)}$.

Examples:

Γ-index theorem: $D\colon C^\infty(X,E) \to C^\infty(X,F)$ elliptic. D_k lift of D to X_k. Atiyah's L^2-index theorem inplies

$$\frac{\operatorname{Ind}(D_k)}{[\Gamma : \Gamma_k]} = \operatorname{Ind}(D) = \operatorname{Ind}_\Gamma(\widetilde{D}).$$

Theorem (Lück, 1994):

$$\lim_{k\to\infty} \frac{b_p(X_k)}{[\Gamma : \Gamma_k]} = b_p^{(2)}(X).$$

Conjecture 1: Let $X = \Gamma \backslash \mathbb{H}^d$ be a closed orineted hyperbolic manifold of dimension $d = 2n + 1$. Let $\{\Gamma_j\}_{j \in \mathbb{N}}$ a tower of normal subgroups of Γ. Then

$$\lim_{j \to \infty} \frac{\log T_{X_j}}{[\Gamma : \Gamma_j]} = \log T_X^{(2)} = \mathrm{vol}(X) \cdot t_{\mathbb{H}^d}^{(2)}.$$

Conjecture 2 (Bergeron, Vankatesh):

$$\lim_{j \to \infty} \frac{\log |H_n(X_j, \mathbb{Z})_{\mathrm{tors}}|}{[\Gamma : \Gamma_j]} = (-1)^n \log T_X^{(2)}.$$