# This Is What I do

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# Introduction I

## I do representations of

- groups
- algebras

## Which kind of groups ?

1. Reductive groups : mainly $p$-adic, often finite, sometimes real
2. Finite or Affine Weyl groups, complex reflection groups
3. Galois groups (more exactly Weil groups)

## Which kind of algebras ?

- (twisted) group algebras of finite groups
- (twisted, extended) affine Hecke algebras
- Lusztig asymptotic Hecke algebras
- $C^*$-algebras

# Spectral extended quotient I

## Definition

- $\Gamma$ finite group acting as automorphisms of a complex affine variety $X$.
- For $x \in X$, let $\Gamma_x$ denote the isotropy group of $x$.
- $\mathrm{Irr}(\Gamma_x)$ set of equivalence classes of irreducible representations of $\Gamma_x$.
- $\widetilde{X} := \{(x,\tau) \,:\, x \in X \;\; \tau \in \mathrm{Irr}(\Gamma_x)\}$.
- $\Gamma$ acts on $\widetilde{X}$ by
$$\gamma' \cdot (x,\tau) := (\gamma' \cdot x, \gamma'_* \tau),$$
where $\gamma'_* \colon \mathrm{Irr}(\Gamma_x) \to \mathrm{Irr}(\Gamma_{\gamma' x})$ is defined by $(\gamma'_*\tau)(\gamma) := \tau(\gamma'^{-1}\gamma\gamma')$.

The spectral extended quotient of $X$ by $\Gamma$ is

$$X/\!/\Gamma := \widetilde{X}/\Gamma.$$

# The ABPS Conjecture I

## Framework

- $G$ quasi-split $p$-adic reductive group
- $\mathrm{Irr}(G)$ set of (isomorphy classes of) irred smooth $G$-repres.
- $L$ Levi subgroup of $G$ and $\sigma \in \mathrm{Irr}(L)$ supercuspidal
- $\mathcal{X}(L)$ group of unramified characters of $L$ (i.e. trivial on every compact subgroup of $L$)
- $\mathfrak{s} = \mathfrak{s}_G = (L, \mathcal{X}(L) \cdot \sigma)_G =: [L, \sigma]_G$
- $\mathrm{Irr}(G)^{\mathfrak{s}} := \{\pi \in \mathrm{Irr}(G) \ : \ \pi$ has supercuspidal support $\mathfrak{s}\}$
- Bernstein decomposition :

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s}} \mathrm{Irr}(G)^{\mathfrak{s}}$$

- The finite group $W_{\mathfrak{s}} := \mathrm{N}_G(\mathfrak{s})/L$ is acting on $T_{\mathfrak{s}} := \mathrm{Irr}(L)^{\mathfrak{s}_L}$ and we can form the spectral extended quotient $T_{\mathfrak{s}}//W_{\mathfrak{s}}$
- $T_{\mathfrak{s}}^{\mathrm{u}}$ set of unitary repres. in $T_{\mathfrak{s}}$

# The ABPS Conjecture II

**Conjecture (A-Baum-Plymen-Solleveld) [now a Theorem in many cases]**

For every $\mathfrak{s}$, there exists a bijection

$$\mu^{\mathfrak{s}} : \mathrm{Irr}(G)^{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$$

which

- restricts to a bijection

$$\mathrm{Irr}(G)^{\mathfrak{s}} \cap \mathrm{Irr}^{\mathrm{t}}(G) \longrightarrow T_{\mathfrak{s}}^{\mathrm{u}}/\!/W_{\mathfrak{s}}$$

  between tempered repres. and the unitary part of $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$,
- is canonical up to permutations within $L$-packets $\Pi_{\varphi}(G)$, i.e., for any Langlands parameter $\varphi$ for $G$, the image of $\Pi_{\varphi}(G) \cap \mathrm{Irr}^{\mathfrak{s}}(G)$ by $\mu^{\mathfrak{s}}$ is canonically defined (assuming the existence of the $\mathrm{LLC}$ for $G$).

# The ABPS Conjecture III

**There is a twisted version for arbitrary $p$-adic reductive groups :**

- "Conjectures about $p$-adic groups and their noncommutative geometry", Contemp. Math., **691**, Amer. Math. Soc., Providence, RI, 2017]
- "Smooth duals of inner forms of $\mathrm{GL}_n$ and $\mathrm{SL}_n$", Doc. Math. **24** (2019).

**Question :**

Does ABPS admits a kind of analogue for real Lie groups ?

**Remark**

A major difference is that, for real groups, two non-associated principal series representations can have a common subquotient. See for instance :

- G. Muić, G. Savin,"The center of the category of $(\mathfrak{g}, K)$-modules", Trans. Amer. Math. Soc. **360** (2008).

# $C^*$-blocks I

**Joint work with A. Afgoustidis :**

**Notation/Definition**

- $G$ a real or $p$-adic reductive group
- $M$ Levi subgroup of $G$
- A character $\chi\colon M \to \mathbb{C}^\times$ is unramified if $\chi$ is trivial on every compact subgroup of $M$
- $\mathcal{X}_{\mathrm{u}}(M)$ group of unitary unramified characters of $M$
- $\omega$ square-integrable irred. repres. of $M$
- $\mathfrak{d} := (M, \mathcal{X}_{\mathrm{u}}(M)\cdot\omega)_G =: [M,\omega]_G$
- $\mathrm{Irr}^{\mathrm{t}}(G)_{\mathfrak{d}} := \{\pi \in \mathrm{Irr}^{\mathrm{t}}(G) \;:\; \pi$ has discrete support $\mathfrak{d}\}$

# $C^*$-blocks II

## The tempered dual $\mathrm{Irr}^{\mathrm{t}}(G)$ :

- There is a decomposition $\mathrm{Irr}^{\mathrm{t}}(G) = \bigsqcup_{\mathfrak{d}} \mathrm{Irr}^{\mathrm{t}}(G)_{\mathfrak{d}}$
- $\mathrm{Irr}^{\mathrm{t}}(G)$ may be identified with the spectrum of $C_{\mathrm{r}}^*(G)$.

## Decomposition of $C_{\mathrm{r}}^*(G)$ [reals Valette (1985), *p*-adics Plymen (1990)] :

$$C_{\mathrm{r}}^*(G) = \bigoplus_{\mathfrak{d}} C_{\mathrm{r}}^*(G; \mathfrak{d}),$$

where $C_{\mathrm{r}}^*(G; \mathfrak{d})$ is a subalgebra of $C_{\mathrm{r}}^*(G)$ with spectrum $\mathrm{Irr}^{\mathrm{t}}(G)_{\mathfrak{d}}$.

## Question

Does the action of $W_{\mathfrak{d}} := \mathrm{N}_G(\mathfrak{d})/M$ on $\mathcal{O} := \mathcal{X}_{\mathrm{u}}(M) \cdot \omega$ : always admit a fixed point ?

- for reals, yes.
- for *p*-adics, not known in general.

# $C^*$-blocks III

## Description $C_r^*(G,\mathfrak{d})$ up to strong Morita equivalence

Joint work with A. Afgoustidis :

- for real Lie groups, we recover Wassermann's Theorem.
- for $p$-adic reductive groups, extending methods of Plymen and his students, we obtain a description, under the hypothesis of the existence of a good fixed-point for the action of $W_{\mathfrak{d}}$ on $\mathcal{O}$.
  A fixed point $\omega$ is good if for each point $\tau \in \mathcal{O}$, the Knapp-Stein decompos. $\mathrm{N}_G(M,\omega)/M = W'_\omega \rtimes R_\omega$ and $\mathrm{N}_G(M,\tau)/M = W'_\tau \rtimes R_\tau$ satisfy :
  - $W'_\tau \subset W'_\omega$,
  - $R_\tau$ is isomorphic with a subgroup of $R_\omega$.

## Theorem [Afgoustidis-A. (2020)]

If $G$ be a quasi-split symplectic, orthogonal or unitary group over a $p$-adic field, then the action of $W_{\mathfrak{d}}$ on $\mathcal{O}$ has always a fixed point.
Characterization of good fixed-points.

# Howe correspondence I

**Definition (Howe)**

It is a correspondence $\Theta$ between irred. repres. of $G$ and $G'$ where $(G, G')$ is a reductive dual pair (for instance $(G, G') = (\mathrm{Sp}_{2n}(F), \mathrm{O}_{2m'}(F))$.

**$\Theta$ has many nice properties (and a vast domain of applications) :**

- It is one-to-one when $F$ is $p$-adic or real (but not for $F$ finite) (For $p$-adics : [Gan-Takeda, "A proof of the Howe duality conjecture", J. Amer. Math. Soc. **29** (2016)]).
- It preserves the Moy-Prasad depth of representations [Pan, Duke Math. J. **113** (2002)].
- It preserves the property of being unipotent when $F = \mathbb{F}_q$ is finite [Adams-Moy, TAMS (1993)].

# Howe correspondence II

**Conjecture (A.-Michel-Rouquier) [Duke Math. J. (1996)].**

Combinatorial description of

$$\pi \mapsto \Theta(\pi) = \{\pi'_1, \pi'_2, \ldots, \pi'_{r'}\}, \quad \text{for } (G, G') = (\mathrm{Sp}_n(\mathbb{F}_q), \mathrm{O}_{2n'}(\mathbb{F}_q)),$$

the validity of which has been established by Pan in arXiv :1901.00623.

**Definition of the $\eta$ correspondence (Gurevich-Howe) [ Progr. Math., 323] and [Jpn. J. Math. 15 (2020)]**

Extraction from $\Theta$ of a one-to-one correspondence $\eta$ in the case of dual pairs $(\mathrm{Sp}_{2n}(\mathbb{F}_q), \mathrm{O}_{N'}(\mathbb{F}_q))$ in "stable range", i.e. such that $N' \leq n$, which is based on a notion of "rank of a representation".

# Howe correspondence III

## Unipotent support (Lusztig) [Adv. Math. 94 (1992)]

Lusztig has attached to any irred. repres. Π of a finite reductive group, a unique rational unipotent class, which has the property that the character of Π is non trivial on it, and has maximal dimension among classes with this property. This class is called the unipotent support of Π.

## Definition of the $\underline{\theta}$ correspondence (A.-Kraskiewicz-Przebinda) [PSPM, AMS (2016)], (J. Epequin Chavez) [J. Algebra 535, 2019]

Extraction from Θ of a one-to-one correspondence $\underline{\theta}$.
If $\pi$ is unipotent, then $\underline{\theta}(\pi)$ has the smallest unipotent support (for the closure order) among the irreducible representations in $\Theta(\pi)$.

## Theorem (Pan) [arXiv :2006.06241]

$\eta$ and $\underline{\theta}$ coincide on their commun domain of definition (i.e. in the stable range case).

# Generalized Springer correspondence I

## Notation

- $G$ a complex (possibly disconnected) reductive group
- $W$ Weyl group of $G$.
- $\mathrm{Unip}(G)$ the unipotent variety of $G$.
- For $u \in G$ unipotent, $A_G(u)$ denotes the component group of the centralizer of $u$ in $G$.

## Enhancement of $\mathrm{Unip}(G)$ :

Let $\mathrm{Unip}_{\mathrm{e}}(G)$ be the set of $G$-conjugacy classes of pairs $(u, \phi)$, with $u \in G$ unipotent and $\phi \in \mathrm{Irr}(A_G(u))$.

## Springer correspondence for $G$ connected (Springer) :

Injective map $\mathrm{Irr}(W) \hookrightarrow \mathrm{Unip}_{\mathrm{e}}(G)$.

# Generalized Springer correspondence II

**Generalized Springer correspondence for $G$ connected [Lusztig, Invent. Math. (1984)]**

Bijection $\bigsqcup_{\mathfrak{c}\in\mathfrak{B}(G)} \mathrm{Irr}(W_{\mathfrak{c}}) \to \mathrm{Unip}_{\mathrm{e}}(G)$ where

- $\mathfrak{C}(G)$ is the set of $G$-conjugacy classes of pairs $(L,(v,\varepsilon))$ such $L$ is Levi subgroup of $G$ and $(v,\varepsilon)\in \mathrm{Unip}_{\mathrm{e}}(L)$ is cuspidal,
- $W_{\mathfrak{c}} := \mathrm{N}_G(\mathfrak{c})/L = \mathrm{N}_G(L)/L$ is a finite Weyl group, where $\mathfrak{c} = (L,(v,\varepsilon))_G$.

**Generalized Springer correspondence for $G$ disconnected [A-Moussaoui-Solleveld, Manuscripta Math. (2018)] :**

Bijection $\bigsqcup_{\mathfrak{c}\in\mathfrak{C}(G)} \mathrm{Irr}\,\mathbb{C}[W_{\mathfrak{c}},\kappa_{\mathfrak{c}}] \to \mathrm{Unip}_{\mathrm{e}}(G)$, where $W_{\mathfrak{c}} := \mathrm{N}_G(\mathfrak{c})/L$, $W_{\mathfrak{c}}^{\circ} := \mathrm{N}_{G^{\circ}}(L^{\circ})/L^{\circ}$ and $\kappa_{\mathfrak{c}}\colon W_{\mathfrak{c}}/W_{\mathfrak{c}}^{\circ}\times W_{\mathfrak{c}}/W_{\mathfrak{c}}^{\circ} \to \mathbb{C}^{\times}$ is a 2-cocycle.

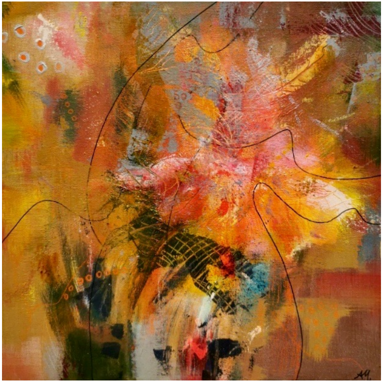
**Remark**

Our motivation : to plug it into the Langlands correspondence.

# Langlands correspondence for $p$-adic groups I

**Several works on the Langlands correspondence for $p$-adic groups :**

- with B. Baum, R. Plymen and M. Solleveld : inner forms of $p$-adic $\mathrm{GL}_n$ and $\mathrm{SL}_n$, principal series of split $p$-adic groups
- with S. Mendes, R. Plymen and M. Solleveld : $\mathrm{SL}_2(F)$, with $F$ of residual char. 2
- with R. Plymen : Weil-restricted $p$-adic groups, how the Moy-Prasad depth changes under $\mathrm{LLC}$
- with A. Moussaoui and M. Solleveld :
  - construction of twisted affine Hecke algebras attached to cuspidal enhanced $L$-parameters of Levi subgroups
  - formulation and proof of a Galois version of the ABPS Conjecture.



Thank you very much for your attention !