Plancherel formulas for reductive groups, symmetric spaces and Whittaker functions

II. Spherical functions and Fourier inversion

Erik van den Ban

Utrecht University

AIM Research Community

Representation Theory and non-commutative geometry

Organizers: P. Clare, N. Higson, B. Speh

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Plancherel identity

Definition Fourier transform

For $f \in C_c^\infty(G/H:\chi)$, $P \in \mathcal{P}_{\mathrm{st}}$, $\sigma \in (M_P)^\wedge_{\chi,ds}$ and $\nu \in i\,\mathfrak{a}_P^*$, the Fourier transform $\hat{f}(P,\sigma,\nu)$ is the element of $(\mathcal{V}^\chi_{\sigma,ds})^* \otimes L^2(K/K_P:\sigma_P)$, defined by

$$\hat{f}(P,\sigma,\nu)(\eta) := \int_{G/H} f(x)\, \pi_{P,\sigma,-\nu}(x)\, j^\circ(P,\sigma,-\nu)(\eta)\; dx$$

Remark: The map $f \mapsto \hat{f}(P,\sigma,\nu)$ intertwines L with $\mathrm{id}_{(\mathcal{V}^\chi_{\sigma,ds})^*} \otimes \pi_{P,\sigma,\nu}$.

Theorem (Plancherel)

$$\|f\|_{L^2}^2 = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim} \; \sum_{\sigma \in (M_P)^\wedge_{\chi,ds}} \int_{i\,\mathfrak{a}_P^*} \|\hat{f}(P,\sigma,\nu)\|^2 \; d\lambda_P(\nu)$$

Strategy

Prove identity on the dense subspace $C^\infty(G/H:\chi)_K$ of K-finite functions. Technical tool: sphericalization.

Let (τ, V_τ) be an arbitrary finite dimensional unitary representation of K. It suffices to prove the result for functions in $(C_c^\infty(G/H:\chi) \otimes V_\tau)^K$.

τ-spherical functions

Definition For X a left K-manifold:

$$\begin{aligned} C^\infty(\tau : X) &:= \{f : X \to V_\tau \mid f(kx) = \tau(k)f(x)\} \\ &\simeq (C^\infty(X) \otimes V_\tau)^K. \end{aligned}$$

Likewise: $C_c^\infty(\tau : G/H : \chi) \simeq (C_c^\infty(G/H : \chi) \otimes V_\tau)^K$.

By triviality on tensor component V_τ, and by using the isometric identification $\iota : \bar{\mathcal{V}}_{\sigma,ds} \to \mathcal{V}^*_{\sigma,ds}$, via inner product, Fourier transform becomes

$$\begin{array}{ccc} C_c^\infty(\tau : G/H : \chi) & \overset{\hat{\cdot}(P,\sigma,\nu)}{\longrightarrow} & \mathcal{V}^*_{\sigma,ds} \otimes L^2(\tau : K/K_P : \sigma_P) \\ & & \downarrow \;\; \iota^{-1} \otimes \mathrm{ev}_e \\ \mathcal{F}_{P,\sigma,\nu} & \searrow & \bar{\mathcal{V}}_{\sigma,ds} \otimes (\mathcal{H}_\sigma \otimes V_\tau)^{K_P} \\ & & \downarrow \;\; \simeq \text{ (matrix coefficient)} \\ & & \oplus_{v \in {}_P\mathcal{W}} L^2_\sigma(\tau_P : M_P/M_P \cap vHv^{-1} : (v\chi)_P) \\ & & =: \mathcal{A}_{2,P,\sigma} \end{array}$$

Notation (HC): $T \mapsto \psi_T$ for composition of vertical maps (isometric).

Assumption: (to simplify exposition) ${}_P\mathcal{W} = \{1\}$ (automatic for group, Riemannian symmetric , complex symmetric, Whittaker case). Then

$$\mathcal{A}_{2,P,\sigma} = L^2_\sigma(\tau_P : M_P/M_P \cap H : \chi_P).$$

Plancherel identity for spherical functions

Definition

$$\begin{aligned}\mathcal{A}_{2,P} &:= \oplus_{\sigma\in(M_P)^\wedge_{\chi,ds}} \mathcal{A}_{2,P,\sigma} \\ &= L^2_{ds}(\tau_P : M_P/M_P\cap H : \chi_P)\end{aligned}$$

Lemma $\mathcal{A}_{2,P}$ is finite dimensional
(gp: HC, ss: Oshima-Matsuki, wh: HC, Wallach).

Define: $\mathcal{F}_P : C_c^\infty(\tau : G/H : \chi) \to C^\infty(i\,\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$ by

$$\mathcal{F}_P(f)(\nu) := \oplus_{\sigma\in(M_P)^\wedge_{\chi,ds}} \mathcal{F}_{P,\sigma,\nu}(f).$$

Plancherel identity is equivalent to

$$\|f\|^2_{L^2} = \sum_{P\in\mathcal{P}_{\rm st}/\sim} \int_{i\,\mathfrak{a}_P^*} \|\mathcal{F}_P f(\nu)\|^2\, d\lambda_P(\nu), \qquad (f\in C_c^\infty(\tau : G/H : \chi)).$$

Normalized Eisenstein, Whittaker integrals

Definition
$E^\circ(P, \psi, \nu) \in C^\infty(\tau : G/H : \chi)$ is defined by the following requirements.

- It is linear in $\psi \in \mathcal{A}_{2,P}$.
- For $\psi = \psi_T$ with $T = \eta \otimes \varphi \in \overline{\mathcal{V}}_\sigma^\chi \otimes L^2(\tau_P : K/K_P : \chi_P)$ it is given as the matrix coefficient

$$E^\circ(P, \psi_T, \nu, x) = \langle\, \varphi, \pi_{P,\sigma,\bar\nu}(x) j^\circ(P, \sigma, \bar\nu)\eta \,\rangle.$$

Remark In the Whittaker case, Harish-Chandra calls this the normalized Whittaker integral, notation $\mathrm{Wh}(P, \psi, \nu, x)$.

Lemma

$$\langle\, \mathcal{F}_P f(\nu), \psi \,\rangle = \int_{G/H} f(x)\overline{E^\circ(P, \psi, -\bar\nu, x)}\, dx = \langle\, f, E^\circ(P, \psi, -\bar\nu) \,\rangle.$$

Lemma $E^\circ(P, \psi, \nu)$ depends meromorphically on $\nu \in {}_{\mathrm{P}}\mathfrak{a}^*_{P\mathbb{C}}$. For generic ν it satisfies the following differential equations

$$R_Z E^\circ(P, \psi, \nu) = E^\circ(P, \underline{\mu}_P(Z, \nu)\psi, \nu), \qquad (Z \in \mathfrak{Z}(\mathfrak{g})).$$

Here $\underline{\mu}_P(Z, \nu) \in \mathrm{End}(\mathcal{A}_{2,P})$ is polynomial in ν, algebra homomorphism in Z.

C-functions, Maass-Selberg relations

Setting: $P, Q \in \mathcal{P}_{\mathrm{st}}$; put

$$W({}_{\mathrm{p}}\mathfrak{a}_Q \mid {}_{\mathrm{p}}\mathfrak{a}_P) := \{\varphi \in \mathrm{Hom}({}_{\mathrm{p}}\mathfrak{a}_P, {}_{\mathrm{p}}\mathfrak{a}_Q) \mid \exists w \in W({}_{\mathrm{p}}\mathfrak{a}) : \varphi = w|_{{}_{\mathrm{p}}\mathfrak{a}_P}\}.$$

Asymptotic behavior There exist unique meromorphic functions $C^\circ_{Q|P}(s, \cdot\,) : {}_{\mathrm{p}}\mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$, for $s \in W({}_{\mathrm{p}}\mathfrak{a}_Q \mid {}_{\mathrm{p}}\mathfrak{a}_P)$, such that for generic $\nu \in i\,{}_{\mathrm{p}}\mathfrak{a}^*_P$ and $a \to \infty$ in ${}_PA^+_Q$,

$$E^\circ(P, \psi, \nu)(kam) \sim \sum_{s \in W({}_{\mathrm{p}}\mathfrak{a}_Q | {}_{\mathrm{p}}\mathfrak{a}_P)} a^{s\nu - \rho_Q}[C^\circ_{Q|P}(s, \nu)\psi](m), \quad (m \in M_P)$$

Maass-Selberg relations $C^\circ_{Q|P}(s, -\bar{\nu})^* C^\circ_{Q|P}(s, \nu)$ indep^t of Q, s.

(gp: HC, ss: vdB, Delorme-Carmona, wh: HC)

The definition of $j^\circ(P, \sigma, \nu)$ is motivated by the following lemma.

Lemma $C^\circ_{P|P}(1, \nu) = \mathrm{id}_{\mathcal{A}_{2,P}}$.

Preparation: We need that

$$A(\bar{P}, P, \sigma, \nu)^* = A(P, \bar{P}, \sigma, -\bar{\nu}).$$

Regularity

Lemma $C^{\circ}_{P|P}(1,\nu) = \mathrm{id}_{\mathcal{A}_{2,P}}$.

Proof For $\mathrm{Re}\,\nu$ sufficiently dominant in ${}_P\mathfrak{a}_P^{*+}$, Langlands' limit formula for matrix coefficients of $\mathrm{Ind}_{\bar{P}}^{G}(\sigma \otimes \bar{\nu})$ gives ($\psi = \psi_T$, $T = \eta \otimes \varphi$), for $a \to \infty$ in ${}_{\mathrm{p}}A_P^{+}$ that

$$\begin{aligned} a^{-\nu+\rho_P} E^{\circ}(P,\psi,\nu,am) &= a^{-\nu+\rho_P} \langle \varphi, \pi_{P,\sigma,\bar{\nu}}(ma) A(\bar{P},P,\sigma,\bar{\nu})^{-1} j(\bar{P},\sigma,\bar{\nu})\eta \rangle \\ &= a^{-\nu+\rho_P} \langle [A(\cdots)^{-1*}\varphi], \pi_{\bar{P},\sigma,\bar{\nu}}(ma) j(\bar{P},\sigma,\bar{\nu})\eta \rangle \\ &\sim \langle A(\cdots)[A(\cdots)^{-1*}\varphi](m), \mathrm{ev}_e j(\bar{P},\sigma,\bar{\nu})\eta \rangle \\ &= \langle \varphi(m), \eta \rangle = \psi(m). \end{aligned}$$

Corollary For $P, Q \in \mathcal{P}_{\mathrm{st}}$, $s \in W({}_{\mathrm{p}}\mathfrak{a}_Q \mid {}_{\mathrm{p}}\mathfrak{a}_P)$,

$$C^{\circ}_{Q|P}(s,-\bar{\nu})^{*} C^{\circ}_{Q|P}(s,\nu) = \mathrm{id}_{\mathcal{A}_{2,P}}.$$

In particular, $C^{\circ}_{Q|P}(s,\nu) \in \mathrm{U}(\mathcal{A}_{2,P},\mathcal{A}_{2,Q})$ for ν imaginary.

Corollary The meromorphic functions $\nu \mapsto C^{\circ}_{Q|P}(s,\nu)$ are regular on $i\,{}_P\mathfrak{a}_P^{*}$.

Remark This implies that $E^{\circ}(P,\psi,\nu)$ is regular for imaginary ν, hence that $j^{\circ}(P,\sigma,\nu)$ is regular for such ν.

Extension to the Schwartz space

Definition (HC Schwartz space)

$\mathcal{C}(G/H:\chi)$ is the space of $f \in C^\infty(G/H:\chi)$ such that

$$w^N L_u f \in L^2(G/H:\chi) \qquad (u \in U(\mathfrak{g}), N \in \mathbb{N}).$$

Here $w(kah) = (1 + |\log a|)$, for $k \in K, a \in {}_{\mathrm{p}}A, h \in H$.

Let $\mathcal{S}(i\,{}_{\mathrm{p}}\mathfrak{a}_P^*)$ denote the usual space of Schwartz functions on the finite dimensional real linear space $i\,{}_{\mathrm{p}}\mathfrak{a}_P^*$.

Theorem For each $P \in \mathcal{P}_{\mathrm{st}}$ the map $\mathcal{F}_P$ is continuous linear

$$\mathcal{C}(\tau : G/H : \chi) \to \mathcal{S}(i\,{}_{\mathrm{p}}\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}.$$

Proof for gp: HC, for ss: vdB, Carmona–Delorme, for wh: vdB.

The following strategy works in all cases.

Extension to the Schwartz space, II

Theorem $\mathcal{F}_P\colon \mathcal{C}(\tau : G/H : \chi) \to \mathcal{S}(i\,{}_{\mathrm{p}}\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$ is cont^s linear.

Strategy of Proof

(a) the generalized vector map $j(\bar{P}, \sigma, \nu)$ is defined for $\mathrm{Re}\nu$ sufficiently P-dominant.

(b) derive a Bernstein-Sato type functional equation for $j(\bar{P}, \sigma, \nu)$.

(c) use (b) to extend $j(\bar{P}, \sigma, \nu)$ meromorphically. Singular set is a locally finite union of real translates of root hyperplanes. Gives estimates for $j(\bar{P}, \sigma, \nu)$ with uniformity for $\mathrm{Re}\nu$ in translates of the cone of P-dominant elements.

(d) get moderate estimates for $E^\circ(P, \sigma, \nu)$ on G/H which are of the type of uniformity mentioned in (c).

(e) improve estimates with uniformity in ν by repeated application of the differential equations coming from $\mathfrak{Z}(\mathfrak{g})$ (inspired by Wallach's technique for fixed ν).

(f) improved estimates are uniformly tempered in the range $\nu \in i\,{}_{\mathrm{p}}\mathfrak{a}_P^*$, hence lead to estimates for $\langle\, \mathcal{F}_P f, \psi \,\rangle = \langle\, f, E^\circ(P, \psi, \nu) \,\rangle$.

Wave packets, Spherical Fourier inversion

Definition For $P \in \mathcal{P}_{\rm st}$ define $\mathcal{W}_P : \mathcal{S}(i\,{}_{\rm p}\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P} \to C^\infty(\tau : G/H : \chi)$ by

$$\mathcal{W}_P(\psi)(x) = \int_{i\,{}_{\rm p}\mathfrak{a}_P^*} E^\circ(P, \psi(\nu), \nu, x)\, d\lambda_P(\nu).$$

Theorem $\mathcal{W}_P$ maps continuously to $\mathcal{C}(\tau : G/H : \chi)$.

(gp: HC, ss: vdB–C–D, wh: vdB).

Proof In all cases: a theory of the constant term with parameters: holomorphic version of HC's functions of type $\mathrm{II}(\lambda)$. Missing argument in Whittaker case.

Lemma The composition $\mathcal{W}_P \mathcal{F}_P$ depends on P through $[P] \in \mathcal{P}_{\rm st}/\sim$

(consequence of Maass-Selberg relations).

Lemma $\mathcal{F}_P$ and $\mathcal{W}_P$ are adjoint.

Since $\|\mathcal{F}_P f\|^2 = \langle\, f, \mathcal{W}_P \mathcal{F}_P f\, \rangle$ the spherical Plancherel identity follows from:

Theorem: spherical Fourier inversion

$$I = \sum_{P \in \mathcal{P}_{\rm st}/\sim} \mathcal{W}_P \mathcal{F}_P \qquad \text{on } \mathcal{C}(\tau : G/H : \chi) \qquad \text{(SFI)}.$$

Final part of the talk: sketch of proof for both ss (vdB–S) and wh (vdB).

Cone supported functions

There exists an open polyhedral cone ${}_{\mathrm{p}}\mathfrak{a}^+$ such that (${}_{\mathrm{p}}A^+ = \exp({}_{\mathrm{p}}\mathfrak{a}^+)$)

$$G_+ := K {}_{\mathrm{p}}A^+ H = K \exp({}_{\mathrm{p}}\mathfrak{a}^+) H \qquad \text{open dense in } G.$$

Cases:

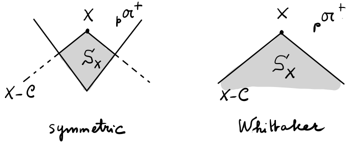
(a) Symmetric space: ${}_{\mathrm{p}}A^+$ is positive chamber for $\Sigma^+(\mathfrak{g}^{\sigma\theta}, {}_{\mathrm{p}}\mathfrak{a})$.

(b) Group: ${}_{\mathrm{p}}\mathfrak{a}^+ = \underline{\mathfrak{a}}^+ \times -\underline{\mathfrak{a}}^+$.

(c) Whittaker: ${}_{\mathrm{p}}A^+ = A$.

Notation

- $\mathcal{C} \subset {}_{\mathrm{p}}\mathfrak{a}$ is the cone dual to ${}_{\mathrm{p}}\mathfrak{a}^+(P_\varnothing)$; $P_\varnothing$ minimal in $\mathcal{P}_{\mathrm{st}}$.
- $C^\infty_{cs}(G/H : \chi)$ is the collection of $f \in C^\infty(G/H : \chi)$ such that there exists a subset of ${}_{\mathrm{p}}\mathfrak{a}$ of the form $S_X := \mathrm{cl}((X - \mathcal{C}) \cap {}_{\mathrm{p}}\mathfrak{a}^+)$ such that $\mathrm{supp} f \subset K \exp(S_X) H$.



Remark For ss: $C^\infty_{cs}(G/H : \chi) = C^\infty_c(G/H)$. For wh: not the case.

Let $P_\varnothing = M_\varnothing A_\varnothing N_\varnothing$ be the minimal element in $\mathcal{P}_{\mathrm{st}}$. Then $M_\varnothing / M_\varnothing \cap H$ is compact, so $\sigma \in \widehat{M}^\chi_{\varnothing, ds} \implies \dim(\sigma) < \infty$.

First step towards proof of (SIF): investigation of $\mathcal{W}_\varnothing \mathcal{F}_\varnothing = \mathcal{W}_{P_\varnothing} \mathcal{F}_{P_\varnothing}$.

Recall:

$$G_+ = K_{\mathrm{p}} A^+ H \qquad \text{open dense in } G.$$

Theorem: There exists unique functions $E_+(\nu) \in \mathcal{A}^*_{2,\varnothing} \otimes C^\infty(\tau : G_+/H : \chi)$ depending meromorphically on $\nu \in {}_{\mathrm{p}}\mathfrak{a}^*_{\mathbb{C}}$ such that, for $\psi \in \mathcal{A}_{2,\varnothing} = \mathcal{A}_{2,P_\varnothing}$,

$$E(P_\varnothing, \psi, \nu)(x) = \sum_{s \in W({}_{\mathrm{p}}\mathfrak{a})} E_+(s\nu, x) C^\circ(s,\nu)(\psi), \qquad (x \in G_+/H).$$

$$E_+(\nu, a)(\psi) = a^{\nu-\rho} \sum_{m \in \mathbb{N}\Sigma^+({}_{\mathrm{p}}\mathfrak{a})} a^{-m} \Gamma_m(\nu)(\psi), \qquad (a \in {}_{\mathrm{p}}A^+).$$

Here $C^\circ(s,\nu) := C^\circ_{P_\varnothing | P_\varnothing}(s,\nu)$, $\Gamma_m(\nu) \in \mathcal{A}^*_{2,\varnothing} \otimes V_\tau$, and $\Gamma_0(\nu)(\psi) = \psi(e)$.

Contour shift à la Helgason (G/K)

For $f \in C_c^\infty(\tau : G/H : \chi)$, $x \in G_+$,

$$\begin{aligned}
\mathcal{W}_\varnothing \mathcal{F}_\varnothing f(x) &= \int_{i\,{}_{\mathfrak{p}}\mathfrak{a}^*} \sum_{s \in W} E_+(s\nu, x) C^\circ(s,\nu) \mathcal{F}_\varnothing f(\nu)\, d\lambda(\nu) \\
&= \sum_{s\in W} \int_{i\,{}_{\mathfrak{p}}\mathfrak{a}^*} E_+(\nu, x) C^\circ(s, s^{-1}\nu) \mathcal{F}_\varnothing f(s^{-1}\nu)\, d\lambda(\nu) \\
&= |W| \int_{i\,{}_{\mathfrak{p}}\mathfrak{a}^*} E_+(\nu, x) \mathcal{F}_\varnothing(f)(\nu)\, d\lambda(\nu) \\
&= |W| \int_{i\,{}_{\mathfrak{p}}\mathfrak{a}^* - \eta} E_+(\nu, x) \mathcal{F}_\varnothing(f)(\nu)\, d\lambda(\nu) + \text{residual integrals} \\
&= \mathcal{T}_\eta f(x) + \mathrm{ResInt}(f),
\end{aligned}$$

with $\eta \in {}_{\mathfrak{p}}\mathfrak{a}^*$ sufficiently $P_\varnothing$-dominant. These residues are picked up along finitely many real translates of root hyperplanes. R_Z acts by $\underline{\mu}(Z,\nu)$ in the integrals on the right. For suitable $Z_0 \in \mathfrak{Z}(\mathfrak{g})$ the residues are cancelled so that

$$R_{Z_0} \mathcal{W}_\varnothing \mathcal{F}_\varnothing f(x) = R_{Z_0} \mathcal{T}_\eta f(x)$$

By sending $\eta \to \infty$ and applying a Paley-Wiener type estimation one concludes, for $f \in C_c^\infty(\tau : G_+/H : \chi)$,

$$\mathrm{supp}(f) \subset K \exp(S_X) H \implies \mathrm{supp} R_{Z_0} \mathcal{W}_\varnothing \mathcal{F}_\varnothing f \subset K \exp(S_X) H.$$

Inversion by a shifted integral

Lemma The operator $R_{Z_0}\mathcal{W}_\varnothing\mathcal{F}_\varnothing \in \mathrm{End}(C_c^\infty(\tau : G_+/H : \chi))$ is support preserving.

Proof: By combining above with symmetry of the operator.

Theorem $R_{Z_0}\mathcal{W}_\varnothing\mathcal{F}_\varnothing = R_{Z_0}$.

Proof:

- The radial part of the operator on the left is essentially a differential operator D on ${}_{\mathrm{p}}A^+$.
- D commutes with the radial parts of all $Z \in \mathfrak{Z}(\mathfrak{g})$.
- coefficients of D satisfy cofinite system of differential equations, which makes that D is determined by its behavior at infinity.
- asymptotically, $D \sim \mathrm{rad}(R_{Z_0})$, hence $D = \mathrm{rad}(R_{Z_0})$.

Theorem For all $f \in C_c^\infty(\tau : G/H : \chi)$ and η sufficiently $P_\varnothing$-dominant, one has

$$f = \mathcal{T}_\eta(f) \qquad \text{on} \quad G_+.$$

Proof:

- Induction $\rightsquigarrow \mathrm{ResInt}(f) \in C^\infty(\tau : G/H : \chi)$, hence $\mathcal{T}_\eta f \in C^\infty(\tau : G/H : \chi)$.
- By Paley-Wiener type estimation, $\mathcal{T}_\eta f \in C_{cs}^\infty(\tau : G/H : \chi)$.
- $\rightsquigarrow f - \mathcal{T}_\eta f \in C_{cs}^\infty(\tau : G/H : \chi)$.
- $f - \mathcal{T}_\eta f$ is annihilated by the analytic linear partial differential operator R_{Z_0}.
- By Holmgren uniqueness, $f - \mathcal{T}_\eta f = 0$.

Identification of Residual integrals

Have found:

$$\mathcal{W}_{P_\varnothing}\mathcal{F}_{P_\varnothing} f = \mathcal{T}_\eta f - \mathrm{ResInt}(f), \qquad \mathcal{T}_\eta f = f.$$

Corollary

$$f = \mathcal{W}_\varnothing \mathcal{F}_\varnothing f + \mathrm{ResInt}(f).$$

One can organize the residue scheme so that it allows induction over *M*-components of parabolic subgroups. By comparison of asymptotic behavior along *A*-components, one can identify:

$$\mathrm{ResInt}(f) = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim, P \neq P_\varnothing} \mathcal{W}_P \mathcal{F}_P f$$

This completes the proof of (SFI), hence of the Plancherel identity.

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