The Plancherel formula for real reductive groups II. The general case

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The Plancherel formula – strategy

- G linear connected semisimple group, $\widehat{G}$ its unitary dual
- Plancherel formula for G equivalent to inversion formula for $f \in C_c^\infty(G)$:

$$f(e) = \int_{\widehat{G}} \Theta_\pi(f)\, d\mu(\pi), \qquad \Theta_\pi(f) = \int_G \Theta_\pi(x) f(x)\, dx.$$

- Θ_π locally integrable and analytic on G_{reg}, conjugation-invariant
  $\rightsquigarrow$ determined by its values on a maximal set of non-conjugate Cartan subalg. $H_1, \ldots, H_r$
- Weyl Integral Formula:

$$\Theta_\pi(f) = \sum_{j=1}^{r} \frac{1}{|W(G,H_j)|} \int_{H_j} \varepsilon_{H_j}(h) \overline{D_{H_j}(h)} \Theta_\pi(h) \cdot F_f^{H_j}(h)\, dh$$

- Strategy: express the orbital integrals $F_f^{H_j}$ in terms of $\Theta_\pi(f)$ (for sufficiently many π), then express $f(e)$ in terms of (derivatives of) $F_f^{H_j}$ at $h = e$

The case $\mathrm{SL}(2,\mathbb{R})$ – revisited

- Two conjugacy classes of Cartan subgroups:
  - $H = MA \rightsquigarrow$ minimal parabolic subgroup $P = MAN \rightsquigarrow$ unitary principal series $\pi_{\sigma,i\lambda}$
  - $T = K \rightsquigarrow$??? $\rightsquigarrow$ discrete series $\pi_n^\pm$
- Weyl integration formula + character formulas:

$$\begin{aligned}\Theta_n(f) &= \int_T (\ldots) F_f^T(t)\,dt + \int_H (\ldots) F_f^H(h)\,dh\\ \Theta_{\pm,i\lambda}(f) &= \int_H (\ldots) F_f^H(h)\,dh\end{aligned}$$

- Using Euclidean Fourier analysis on H and Fourier series on T + singularities of F_f^T $\rightsquigarrow$ solve for $F_f^T(k_\theta)$ (or rather $\frac{d}{d\theta}\Big|_{\theta=0} F_f^T(k_\theta) = -2\pi i f(e)$)

First: Understand the discrete series!

Discrete series – existence

For a real reductive group G let $\widehat{G}_{\mathrm{ds}} \subseteq \widehat{G}$ denote the equivalence classes of discrete series.

Questions

- When is $\widehat{G}_{\mathrm{ds}} \neq \emptyset$?
- How to parameterize $\widehat{G}_{\mathrm{ds}}$?
- Character formulas?

Let G be a connected semisimple Lie group with finite center, $K \subseteq G$ a maximal compact subgroup and $T \subseteq K$ a maximal torus (= Cartan subgroup).

Theorem (Harish-Chandra)

The following are equivalent:

- $\widehat{G}_{\mathrm{ds}} \neq \emptyset$
- T is a Cartan subgroup of G
- $\mathrm{rank}(G) = \mathrm{rank}(K)$

Examples: $G = \mathrm{SL}(n, \mathbb{R})$ has rank $n-1$, $K = \mathrm{SO}(n)$ has rank $\lfloor \frac{n}{2} \rfloor$, so $\widehat{G}_{\mathrm{ds}} \neq \emptyset \Leftrightarrow n = 2$

Discrete series – parameterization

- Assume that G is linear connected semisimple and that a maximal torus $T \subseteq K$ is also a Cartan subgroup for G. $\Rightarrow \widehat{G}_{\mathrm{ds}} \neq \emptyset$
- Let $\Sigma_K = \Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \subseteq \Sigma_G = \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and denote by W_K the Weyl group of Σ_K

Theorem

For every $\lambda \in (i\mathfrak{t})^*$ (*Harish-Chandra parameter*) satisfying

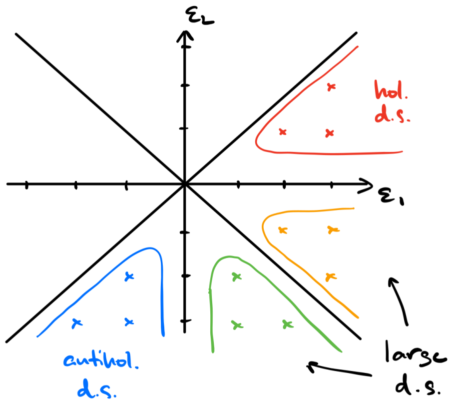
- (non-singular) $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta_G$,
  $\rightsquigarrow \Sigma_G^+ = \{\alpha \in \Sigma_G : \langle \lambda, \alpha \rangle > 0\}, \Sigma_K^+ = \Sigma_G^+ \cap \Sigma_K$, ρ_G, ρ_K half-sums of positive roots
- (analytically integral) $\lambda + \rho_G$ integrates to a character of T

there exists a unique discrete series representation π_λ with lowest K-type of highest weight $\Lambda = \lambda + \rho_G - 2\rho_K$ with respect to Σ_K^+ (*Blattner parameter*).
Moreover, $\pi_\lambda \simeq \pi_\mu$ iff $\lambda = w\mu$ for some $w \in W_K$.
This construction exhausts $\widehat{G}_{\mathrm{ds}}$.

Discrete series – example $G = \mathrm{Sp}(2, \mathbb{R})$

Let $G = \mathrm{Sp}(2, \mathbb{R})$, $K = \mathrm{U}(2)$, $T = \mathrm{U}(1) \times \mathrm{U}(1)$, then

- $\Sigma_G = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}$
  $\Sigma_K = \{\pm(\varepsilon_1 - \varepsilon_2)\}$, $W_K = \mathbb{Z}_2$,
- $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2$ non-singular
  $\Leftrightarrow \lambda_1, \lambda_2, \lambda_1 \pm \lambda_2 \neq 0$
- $\lambda + \rho_G$ analytically integral $\Leftrightarrow \lambda_1, \lambda_2 \in \mathbb{Z}$
- 4 families of discrete series:
  - holomorphic discrete series
  - antiholomorphic discrete series
  - 2 "large" discrete series



Discrete series – characters

General: $\pi \in \widehat{G} \rightsquigarrow \Theta_\pi \in \mathcal{D}'(G)$ *invariant eigendistribution*, i.e.

- conjugation-invariant $\rightsquigarrow$ reduce to Cartan subalgebras H
- $\mathcal{Z}(\mathfrak{g}_\mathbb{C}) \subseteq \mathcal{U}(\mathfrak{g}_\mathbb{C})$ acts by scalars. $\rightsquigarrow$ differential equations on each H

Theorem (Harish-Chandra)

Let $\Theta \in \mathcal{D}'(G)$ be an invariant eigendistribution, then Θ is locally integrable on G and analytic on G^{reg}. On each connected component of H^{reg}, H a Cartan subgroup, it can be written as

$$\Theta(h) = \frac{\tau_H(h)}{D_H(h)},$$

with $D_H(h)$ the Weyl denominator (given in terms of $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$) and

$$\tau_H(h\exp(X)) = \sum_{w \in W(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})} p_w(X) e^{(w\lambda)(X)} \qquad (X \in \mathfrak{h}),$$

where the p_w are polynomials on $\mathfrak{h}$ and λ is related to the eigenvalues of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ ($\leftrightarrow$ inf. char. of π for $\Theta = \Theta_\pi$). Moreover, $\varepsilon_H(h)\tau_H(h)$ is continuous on H ($\varepsilon_H(h) \in \{\pm 1\}$ in terms of roots).

Discrete series – characters

Let Θ_λ denote the character of the discrete series representation π_λ, $\lambda \in (i\mathfrak{t})^*$.

Theorem

On the compact Cartan T, the numerator τ_T of Θ_λ is given by

$$\tau_T(\exp(X)) = (-1)^{\frac{1}{2}\dim(G/K)} \sum_{w \in W_K} \det(w) e^{(w\lambda)(X)} \qquad (X \in \mathfrak{t}).$$

Proof: Show that Θ_λ agrees with the character of $\pi_\lambda|_K$ which can be expressed in terms of the K-types of π_λ. Use the information about the lowest K-type of π_λ and Weyl group action. □

Example $G = \mathrm{SL}(2, \mathbb{R})$

$T = K = \mathrm{SO}(2) \subseteq G$ and $\Sigma_G = \{\pm\alpha\}$, $\Sigma_K = \emptyset$, $W_K = \{e\}$
$\lambda = \frac{n}{2}\alpha$ non-singular and $\lambda + \rho_G$ analytically integral $\Leftrightarrow n \in \mathbb{Z} \setminus \{0\}$

$$D_T(k_\theta) = \mathrm{sgn}(n)(e^{i\theta} - e^{-i\theta}), \quad \tau_T(k_\theta) = -e^{in\theta} \qquad \Rightarrow \Theta_n(k_\theta) = -\mathrm{sgn}(n)\frac{e^{in\theta}}{e^{i\theta} - e^{-i\theta}}.$$

Now: Contribution of an arbitrary Cartan subgroup $H \subseteq G$ to $L^2(G)$

Cuspidal parabolic subgroups

Let H be a θ-stable Cartan subgroup, then $H = TA$ with $T = H \cap K$ and $A = H \cap \exp \mathfrak{p}$.
$\rightsquigarrow$ parabolic subgroup $P = MAN$ with $T \subseteq M$ compact Cartan subgroup
$\rightsquigarrow$ M has discrete series

Definition (cuspidal parabolic subgroup)

A parabolic subgroup $P = MAN \subseteq G$ is called *cuspidal* if M has discrete series.

$$\{\text{cuspidal parabolic subgroups}\}/\sim \quad \longleftrightarrow \quad \{\text{Cartan subalgebras}\}/\sim$$

Examples

- minimal parabolic subgroups $\leftrightarrow$ Cartan subalgebras $H = TA$ with $\dim A$ maximal
- $G = MAN$ with $M = G$, $A = N = \{e\}$ iff G has discrete series $\leftrightarrow$ $T \subseteq K \subseteq G$ max. torus
- For $G = GL(n, \mathbb{R})$: parabolic subgroups $P = MAN$, $MA \simeq \mathrm{GL}(n_1, \mathbb{R}) \times \ldots \times \mathrm{GL}(n_r, \mathbb{R})$.
  P cuspidal $\Leftrightarrow$ $n_1, \ldots, n_r \in \{1, 2\}$

Principal series representations

Let $P = MAN \subseteq G$ be a cuspidal parabolic subgroup. For $\sigma \in \widehat{M}_{\mathrm{ds}}$ and $\lambda \in \mathfrak{a}^*$ we form the unitary principal series representation

$$\pi_{\sigma,i\lambda} = \mathrm{Ind}_P^G(\sigma \otimes e^{i\lambda} \otimes 1).$$

Note: Since $\widehat{M}_{\mathrm{ds}}$ is parameterized by non-singular characters of a compact Cartan $T \subseteq M$ and $e^{i\lambda}$ runs through the unitary characters of A, the representations $\pi_{\sigma,i\lambda}$ are parameterized by "non-singular" unitary characters of the corresponding Cartan subgroup $H = TA \simeq T \times A$.

$\rightsquigarrow$ Langlands classification

Next: Character $\Theta_{\sigma,i\lambda}$ of $\pi_{\sigma,i\lambda}$

Principal series characters – vanishing

Lemma

Let $H = TA$ be a Cartan subgroup of G with corresponding cuspidal parabolic subgroup $P = MAN$ and let $\Theta_{\sigma,i\lambda}$ be the character of the principal series representation $\pi_{\sigma,i\lambda}$ induced from $\sigma \in \widehat{M}_{\mathrm{ds}}$ and $\lambda \in \mathfrak{a}^*$. If a Cartan subgroup $H' \subseteq G$ is not conjugate (within G) to a Cartan subgroup of MA, then

$$\Theta_{\sigma,i\lambda}|_{(H'_{\mathrm{reg}})^G} = 0.$$

$\rightsquigarrow$ partial ordering on the conjugacy classes of Cartan subgroups:

$$H_1 \leq H_2 \quad :\Leftrightarrow \quad H_1 \text{ conjugate to a subgroup of } M_2A_2,$$

where $M_2A_2N_2$ is a parabolic subgroup associated to $H_2 = T_2A_2$.

Let $H_1, \ldots, H_r$ representatives of the conjugacy classes of Cartan subgroups of G, ordered in such a way that $H_i \leq H_j$ implies $i \leq j$.

$\rightsquigarrow$ By the Weyl Integral Formula: $\Theta^{P_j}_{\sigma,i\lambda}(f) = \int_{H_1}(\ldots)F_f^{H_1}(h)\,dh + \cdots + \int_{H_j}(\ldots)F_f^{H_j}(h)\,dh$

$\rightsquigarrow$ triangular system expressing $\Theta^{P_j}_{\sigma,i\lambda}(f)$ in terms of $F_f^{H_i}$

Example: $G = \mathrm{Sp}(2,\mathbb{R})$

Four conjugacy classes of Cartan subgroups/cuspidal parabolic subgroups:

- $T \simeq \mathrm{U}(1) \times \mathrm{U}(1) \rightsquigarrow G$
- $H_1 \simeq \mathrm{U}(1) \times \mathbb{R}^\times \rightsquigarrow P_1 = M_1A_1N_1$ with $M_1 = \mathrm{O}(1) \times \mathrm{Sp}(1,\mathbb{R})$, $N_1 = \mathbb{C} \ltimes \mathbb{R}$ Heisenberg
- $H_2 \simeq \mathbb{C}^\times \rightsquigarrow P_2 = M_2A_2N_2$ with $M_2A_2 = \mathrm{GL}(2,\mathbb{R})$, $N_2 = \mathrm{Sym}(2,\mathbb{R})$
- $H = MA \simeq \mathbb{R}^\times \times \mathbb{R}^\times \rightsquigarrow P = MAN$ minimal, $MA = \mathrm{GL}(1,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})$

$\rightsquigarrow H \leq H_1, H_2 \leq T$

$$\begin{aligned}
\Theta^H(f) &= \int_H (\ldots) F_f^H, \\
\Theta^{H_1}(f) &= \int_H (\ldots) F_f^H + \int_{H_1} (\ldots) F_f^{H_1} \\
\Theta^{H_2}(f) &= \int_H (\ldots) F_f^H + \int_{H_2} (\ldots) F_f^{H_2} \\
\Theta^T(f) &= \int_H (\ldots) F_f^H + \int_{H_1} (\ldots) F_f^{H_1} + \int_{H_2} (\ldots) F_f^{H_2} + \int_T (\ldots) F_f^T
\end{aligned}$$

$\rightsquigarrow$ What is $(\ldots)$?

Principal series characters – reduction

The character $\Theta_{\sigma,i\lambda}$ of $\pi_{\sigma,i\lambda}$ can be expressed in terms of λ and the character Θ_σ of σ:

Induced Characters

For $f \in C_c^\infty(G)$ let

$$f^{(P)}(ma) = a^\rho \int_K \int_N f(kmank^{-1})\,dk\,dn,$$

then

- $\Theta_{\sigma,i\lambda}(f) = (\Theta_\sigma \otimes e^{i\lambda})(f^{(P)})$
- $e^{\rho H(h)}\xi_{-\delta}(h)F_f^{G/H}(h) = \xi_{-\delta_M}(h)F_{f^{(P)}}^{MA/H}(h)$

$\rightsquigarrow$ reduction arguments from G to MA possible
$\rightsquigarrow$ formulate (and show) all statements for G in Harish-Chandra class

Recovering $f(e)$ from $F_f^H(h)$

Examples

- $G = \mathrm{SL}(2,\mathbb{R})$: $f(e) = \text{const} \times \lim_{\theta\to 0} \frac{d}{d\theta} F_f^T(k_\theta)$ and $\frac{d}{dt}\Big|_{t=0} F_f^H(a_t) = 0$.
- $G = \mathrm{SL}(2,\mathbb{C})$: $f(e) = \text{const} \times \partial(\overline{\alpha})\partial(\alpha)F_f^H(e)$, where $\Sigma^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}) = \{\alpha,\overline{\alpha}\}$

For a Cartan subgroup H with Lie algebra $\mathfrak{h}$ let $\Sigma_G^+ \subseteq \Sigma_G = \Sigma(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ a positive system within the root system of $(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$. For each $\alpha \in \Sigma_G^+$ let

$$\partial(\alpha)F(h) = \frac{d}{dt}\Big|_{t=0} F(h\exp(t\,\mathrm{Re}\,H_\alpha)) + i\,\frac{d}{dt}\Big|_{t=0} F(h\exp(t\,\mathrm{Im}\,H_\alpha)),$$

where $H_\alpha \in \mathfrak{h}_\mathbb{C}$ corresponds to $\alpha \in \mathfrak{h}_\mathbb{C}^*$ via the Killing form. Let

$$\partial(\omega_H) = \prod_{\alpha\in\Sigma_G^+} \partial(\alpha).$$

Theorem

$$\partial(\omega_H)F_f^H(e) = c_H f(e)$$

with $c_H = 0$ unless $H = TA$ with $\dim T$ maximal, and $c_H \neq 0$ in that case.

Properties of F_f^H

Let $f \in C_c^\infty(G)$ and $H \subseteq G$ a Cartan subgroup.

- $F_f^H \in C^\infty(H_{\text{reg}})$, $\operatorname{supp} F_f^H \subseteq H$ compact
- $F_f^H \in C^\infty(H_{\text{reg}}^*)$, where $H_{\text{reg}}^* = \{h \in H : h^\alpha \neq 1 \text{ for all non-cpt. imaginary } \alpha \in \Sigma_G\} \supseteq H_{\text{reg}}$
- F_f^H extends to a smooth function on the closure of each connected component of H_{reg}^*.
- If two Cartan subgroups H_T and H_A satisfy $\mathfrak{h}_T = \mathfrak{h}' \oplus \mathfrak{t}$ and $\mathfrak{h}_A = \mathfrak{h}' \oplus \mathfrak{a}$ with $\mathfrak{t}, \mathfrak{a} \subseteq \mathfrak{sl}(2,\mathbb{R}) \subseteq \mathfrak{g}$ as before, then for each "semiregular" element $h_0 \in H_T \cap H_A$:

$$F_f^{H_T}(h_0)^+ - F_f^{H_T}(h_0)^- = \text{const} \times F_f^{H_A}(h_0).$$

Strategy (very rough)

- Let $H_1, \dots, H_r$ be representatives of the conjugacy classes of Cartan subgroups of G and $P_1, \dots, P_r$ corresponding cuspidal parabolic subgroups. Assume H_1 is maximally compact.
- Apply the Weyl Integration Formula and the character formulas for $\Theta^{P_1}_{\sigma,i\lambda}|_{H_i} = \frac{\tau_{H_i}}{D_{H_i}}$ to write

$$\Theta^{P_1}_{\sigma,i\lambda}(f) = \sum_{i=1}^{r} \pm \int_{H_i} \varepsilon_{H_i}(h)\tau_{H_i}(h)F^{H_i}_f(h)\,dh$$

- Use $\partial(\omega_{H_i})[\varepsilon_{H_i}\tau_{H_i}] = p(\sigma, i\lambda)[\varepsilon_{H_i}\tau_{H_i}]$ in each of the terms and integrate by parts, taking into account the jumps of $F^{H_i}_f$ $\rightsquigarrow$ total effect of discontinuities drops out

$$\rightsquigarrow p(\sigma, i\lambda)\Theta^{P_1}_{\sigma,i\lambda}(f) = \sum_{i=1}^{r} \pm \int_{H_i} \varepsilon_{H_i}(h)\tau_{H_i}(h)\cdot\partial(\omega_{H_i})F^{H_i}_f(h)\,dh$$

- Use character formula for $\varepsilon_{H_1}(h)\tau_{H_1}(h)$ and Fourier analysis on $H_1 = T_1 \times A_1$ to sum and integrate over σ and λ and the fact that $\partial(\omega_{H_1})F^{H_1}_f(e) = c_{H_1}f(e)$ to eliminate $F^{H_1}_f$
- Continue with H_2, using the reduction from F^{G/H_2}_f to $F^{M_2A_2/H_2}_f$ ...

The Plancherel formula

Plancherel formula

Let G be a linear connected real reductive group, then

$$L^2(G) \simeq \bigoplus_{\substack{P=MAN\\ \text{cuspidal}/\sim}} \bigoplus_{\sigma\in\widehat{M}_{\text{ds}}} \int_{\mathfrak{a}_+^*} \pi_{\sigma,i\lambda} \otimes \pi_{\sigma,i\lambda}^* \, d\lambda.$$

More precisely, there exist (explicitly computable elementary analytic) functions $q_P(\sigma,\lambda)$ such that

$$f(e) = \sum_{\substack{P=MAN\\ \text{cuspidal}/\sim}} \sum_{\sigma\in\widehat{M}_{\text{ds}}} \int_{\mathfrak{a}^*} \Theta_{\sigma,i\lambda}(f)\, q_P(\sigma,\lambda)\, d\lambda \qquad (f \in C_c^\infty(G)).$$

Remark: The inversion formula actually holds for all f in Harish-Chandra's Schwartz space $\mathcal{C}(G)$ and all ingredients ($\Theta_\pi(f)$, F_f^H, $f^{(P)}$, ...) are defined for $f \in \mathcal{C}(G)$.
$\rightsquigarrow$ necessary in order to apply Θ_π to matrix coefficients of discrete series

Application: Riemannian symmetric spaces

$$X = G/K \rightsquigarrow L^2(X) \simeq L^2(G)^K \simeq \bigoplus_{\substack{P=MAN \\ \text{cuspidal}/\sim}} \bigoplus_{\sigma \in \widehat{M}_{\mathrm{ds}}} \int_{\mathfrak{a}_+^*} \pi_{\sigma,i\lambda} \otimes (\pi_{\sigma,i\lambda}^*)^K, d\lambda$$

Observation: $\pi_{\sigma,i\lambda}^K \neq \{0\} \Leftrightarrow P = P_{\min}$ and $\sigma = \mathbf{1}$ trivial representation

$$L^2(G/K) \simeq \int_{\mathfrak{a}_+^*} \pi_{\mathbf{1},i\lambda}\, d\lambda \qquad \text{and} \qquad f(e) = \int_{\mathfrak{a}^*} \Theta_{\mathbf{1},i\lambda}(f) \frac{d\lambda}{|c(\lambda)|^2},$$

where

$$c(\lambda) = c_0 \prod_{\alpha \in \Sigma_{G,0}^+} \frac{2^{-i\frac{\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle}} \Gamma(i\frac{\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle})}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + i\frac{\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle}))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + i\frac{\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle}))}.$$

Note: $L^2(G/H) \simeq L^2(G)^H$ is *not* true for H non-compact! $\rightsquigarrow$ ask Erik van den Ban!

Extra: explicit realization of representations

The construction of $\pi_{\sigma,i\lambda} = \mathrm{Ind}_P^G(\sigma \otimes e^{i\lambda} \otimes 1)$ from σ is very explicit (e.g. on L^2-sections of a Hermitian vector bundle $\mathcal{V}_{\sigma,i\lambda}$ over G/P).

Question

How can discrete series representations be constructed?

There are different constructions of the discrete series:

- (G Hermitian, i.e. $\mathfrak{k} = \mathfrak{u}(1) \oplus \mathfrak{k}'$) The (anti-)holomorphic discrete series representations can be realized on Hilbert spaces of (anti-)holomorphic functions on the bounded symmetric domain G/K on which G acts (anti-)holomorphically.
  Example: $\mathrm{SU}(1,1)$ acting on $D = \{z \in \mathbb{C} : |z| < 1\}$
- (Langlands, Schmid) Dolbeaut cohomology $H^k(G/T, \mathcal{L}_{\lambda-\rho})$ of line bundles over G/T
- (Zuckerman) Derived functor modules $A_{\mathfrak{q}}(\lambda)$
- (Parthasarathy, Atiyah–Schmid) Kernel of a Dirac operator on a vector bundle over G/K

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