The Plancherel formula for real reductive groups
I. Examples

Jan Frahm (Aarhus University)

AIM RTG – Language School "Representation Theory for C^*-Theorists"
August 23, 2021

AARHUS
UNIVERSITET
DEPARTMENT OF MATHEMATICS

General theory

Let G be a real reductive Lie group (e.g. $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{O}(p,q)$).
Let dx denote a Haar measure on G, then $G \times G$ acts unitarily on $L^2(G) = L^2(G, dx)$ by

$$[(g,h)\cdot f](x) = f(g^{-1}xh) \qquad (g,h,x \in G).$$

Question

How does $L^2(G)$ decompose into *irreducible* representations of $G \times G$?

- Denote by $\widehat{G}$ the *unitary dual* of G endowed with the *Fell topology*.
- *Generalized Fourier coefficients:* For $f \in C_c^\infty(G)$ and $(\pi, \mathcal{H}_\pi) \in \widehat{G}$:

$$\pi(f) = \int_G f(x)\pi(x)\,dx : \mathcal{H}_\pi \to \mathcal{H}_\pi.$$

Then $\pi(f)$ is trace class, in particular $\pi(f) \in \mathrm{HS}(\mathcal{H}_\pi) \simeq \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*$, and

$$\pi((g,h)\cdot f) = \pi(g)\circ\pi(f)\circ\pi(h)^{-1}$$

$\rightsquigarrow$ $f \mapsto \pi(f)$ intertwines the actions of $G \times G$ on $C_c^\infty(G)$ and $\pi \otimes \pi^*$.
- *Fourier transform:* $C_c^\infty(G) \to \prod_{\pi \in \widehat{G}} \mathrm{HS}(\mathcal{H}_\pi)$, $f \mapsto \widehat{f}$, where $\widehat{f}(\pi) = \pi(f)$

General theory

Theorem

There exists a unique Radon measure μ on $\widehat{G}$ such that

$$\|f\|_{L^2(G)}^2 = \int_{\widehat{G}} \|\pi(f)\|_{\mathrm{HS}(\mathcal{H}_\pi)}^2 \, d\mu(\pi) \qquad (f \in C_c^\infty(G)).$$

In other words, the unitary representation of $G \times G$ on $L^2(G)$ decomposes into the direct integral

$$L^2(G) \simeq \int_{\widehat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \, d\mu(\pi).$$

Goal

Determine the *Plancherel measure* μ explicitly. $\rightsquigarrow$ Harish-Chandra '76

Assumptions on G: *Harish-Chandra class* (closed under passing to Levi subgroups)

(Some results in this talk are for simplicity stated under stronger assumptions.)

Why $\pi \otimes \pi^*$? Why multiplicity one?

A general irreducible unitary representation of $G \times G$ is of the form $\pi \otimes \tau$ with $\pi, \tau \in \widehat{G}$.

Question

Why do only representations of the form $\pi \otimes \pi^*$ occur in $L^2(G)$? Why with multiplicity one?

- Every embedding $\iota : \pi^\infty \otimes \tau^\infty \hookrightarrow C^\infty(G)$ gives rise to $\eta = \delta_e \circ \iota \in \mathrm{Hom}(\pi^\infty \otimes \tau^\infty, \mathbb{C})$:
$$\eta(v \otimes w) = \iota(v \otimes w)(e).$$
- If ι is $G \times G$-equivariant, the embedding is given by taking matrix coefficients:
$$\iota(v \otimes w)(g) = \eta(\pi(g)^{-1} v \otimes w) = \eta(v \otimes \tau(g) w)$$
and hence
$$\eta \in \mathrm{Hom}_G(\pi^\infty \otimes \tau^\infty, \mathbb{C}) \neq \{0\} \quad \overset{\pi,\tau \text{ irred.}}{\Longleftrightarrow} \quad \tau \simeq \pi^*.$$
- $\dim \mathrm{Hom}_G(\pi^\infty \otimes \pi^{*,\infty}, \mathbb{C}) = 1$ (Schur's Lemma) $\Rightarrow$ multiplicity one
- Alternative interpretation: $G \simeq (G \times G)/\mathrm{diag}(G)$ and $\eta \in (\pi \otimes \tau)^{-\infty, \mathrm{diag}(G)}$
$\rightsquigarrow$ generalization to homogeneous spaces $\mathbf{G}/\mathbf{H}$ and $\Pi \in \widehat{\mathbf{G}}$ with $\eta \in \Pi^{-\infty, \mathbf{H}}$

Examples

1. G compact: Peter–Weyl Theorem

$$\int_G |f(x)|^2\,dx = \sum_{[\pi]\in\widehat{G}} d_\pi \|\pi(f)\|^2_{\mathrm{HS}(\mathcal{H}_\pi)} \qquad \text{with } d_\pi = \dim \mathcal{H}_\pi.$$

   Special case: $G = \mathbb{T}$ Fourier series

2. $G = \mathbb{R}$: Fourier transform

$$\int_{\mathbb{R}} |f(x)|^2\,dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2\,d\xi.$$

3. Today: $G = \mathrm{SL}(2,\mathbb{R}), \mathrm{SL}(2,\mathbb{C})$

Characters

Lemma

The inversion formula implies the Plancherel formula:

$$h(e) = \int_{\widehat{G}} \operatorname{tr}(\pi(h))\, d\mu(\pi) \quad (h \in C_c^\infty(G)) \quad \Rightarrow \quad \|f\|_{L^2(G)}^2 = \int_{\widehat{G}} \|\pi(f)\|^2\, d\mu(\pi) \quad (f \in C_c^\infty(G)).$$

Proof: Let $h = f^* * f$ with $f^*(x) = \overline{f(x^{-1})}$, then $h(e) = \|f\|^2$ and $\pi(h) = \pi(f)^*\pi(f)$. □

Definition (distribution character)

$\Theta_\pi(h) = \operatorname{tr}(\pi(h))$ $(h \in C_c^\infty(G))$ defines the distribution character $\Theta_\pi \in \mathcal{D}'(G)$ of π.

- determines π uniquely
- conjugation-invariant
- (Harish-Chandra) locally integrable function, also denoted by $\Theta_\pi(x)$, analytic on the open dense subset of regular elements $G_{\mathrm{reg}} = \{x \in G : \dim Z_G(x) \text{ smallest possible}\}$

$$\rightsquigarrow f(e) = \int_{\widehat{G}} \Theta_\pi(f)\, d\mu(\pi), \qquad \Theta_\pi(f) = \int_G \Theta_\pi(x) f(x)\, dx.$$

Strategy

$$f(e) = \int_{\widehat{G}} \Theta_\pi(f)\, d\mu(\pi) \qquad \Theta_\pi(f) = \int_G \Theta_\pi(x) f(x)\, dx$$

(Very rough) Strategy

Compute $\Theta_\pi(f)$ for all/sufficiently many representations $\pi \in \widehat{G}$ and recover $f(e)$ from $(\Theta_\pi(f))_\pi$.

Problem

$\widehat{G}$ not classified for most real reductive groups G

$\rightsquigarrow$ identify those representations $\pi \in \widehat{G}$ that are contained in $\operatorname{supp} \mu$

$\rightsquigarrow$ $\widehat{G}_{\text{temp}}$: *tempered dual*

The case $G = \mathrm{SU}(2)$

To illustrate the general method, we prove the inversion formula in the case $G = \mathrm{SU}(2)$ (method works essentially in the same way for all compact Lie groups modulo technicalities)

For $G = \mathrm{SU}(2)$: $\widehat{G} = \{[\pi_n] : n \in \mathbb{N}\}$, $\dim \pi_n = n + 1$, $\Theta_n = \Theta_{\pi_n}$

Theorem (Peter–Weyl)

$$f(e) = \sum_{n=0}^{\infty} (n+1)\Theta_n(f)$$

To compute

$$\Theta_n(f) = \int_G \Theta_n(x) f(x)\, dx$$

we first need an expression for the character $\Theta_n(x) = \mathrm{tr}(\pi_n(x))$.
Note: Θ_n is conjugation-invariant and every element in SU(2) is conjugate to a diagonal matrix

$$t_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

$\rightsquigarrow$ Θ_n determined by its values on the *maximal torus* $T = \{t_\theta : \theta \in \mathbb{R}\}$

The case $G = \mathrm{SU}(2)$ – cont'd

Direct computation:

$$\Theta_n(t_\theta) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}$$

To integrate over $G = \{gtg^{-1} : g \in G, t \in T\}$ we use the Weyl integration formula:

$$\int_G \varphi(x)\,dx = \frac{1}{2}\int_T \int_{G/T} \varphi(gtg^{-1})\,d(gT)|D_T(t)|^2\,dt,$$

where $D_T(t_\theta) = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.

$$\Rightarrow \quad \Theta_n(f) = \frac{1}{2}\int_T \underbrace{\overline{D_T(t)}\Theta_n(t)}_{=-(e^{i(n+1)\theta}-e^{-i(n+1)\theta})} \underbrace{D_T(t)\int_{G/T} f(gtg^{-1})\,d(gT)}_{F_f^T(t):=}\,dt$$

$F_f^T \in C^\infty(T)$ is called *orbital integral* of f along T

The case $G = \mathrm{SU}(2)$ – cont'd

To recover $f(e)$ from $(\Theta_n(f))_n$ observe that

$$\begin{aligned}\frac{d}{d\theta}\Big|_{\theta=0} F_f^T(t_\theta) &= 2i\cos\theta \int_{G/T} f(g t_\theta g^{-1})\,d(gT)\Big|_{\theta=0} + 2i\sin\theta \frac{d}{d\theta}\int_{G/T} f(g t_\theta g^{-1})\,d(gT)\Big|_{\theta=0} \\ &= 2if(e)\end{aligned}$$

$\rightsquigarrow$ Recover $\frac{d}{d\theta}\Big|_{\theta=0} F_f^T(t_\theta)$ from

$$\Theta_n(f) = -\frac{1}{2}\int_0^{2\pi} (e^{i(n+1)\theta} - e^{-i(n+1)\theta}) F_f^T(t_\theta)\,\frac{d\theta}{2\pi}$$

$\rightsquigarrow$ Multiply by $(n+1)$, write $(n+1)(e^{i(n+1)\theta} - e^{-i(n+1)\theta}) = i\frac{d}{d\theta}(e^{i(n+1)\theta} + e^{-i(n+1)\theta})$ and integrate by parts:

$$\begin{aligned}(n+1)\Theta_n(f) &= \frac{i}{2}\int_0^{2\pi} \frac{d}{d\theta}(e^{i(n+1)\theta} + e^{-i(n+1)\theta}) F_f^T(t_\theta)\,\frac{d\theta}{2\pi} \\ &= \frac{1}{2i}\int_0^{2\pi} (e^{i(n+1)\theta} + e^{-i(n+1)\theta}) \frac{d}{d\theta} F_f^T(t_\theta)\,\frac{d\theta}{2\pi}.\end{aligned}$$

The case $G = \mathrm{SU}(2)$ – summary

To extract $\frac{d}{d\theta}\Big|_{\theta=0} F_f^T(t_\theta)$, we sum over n and use the Fourier series expansion:

$$\sum_{n=0}^{\infty}(n+1)\Theta_n(f) = \frac{1}{2i}\sum_{m\in\mathbb{Z}} \widehat{\frac{d}{d\theta}F_f^T}(m) = \frac{1}{2i}\,\frac{d}{d\theta}\Big|_{\theta=0} F_f^T(t_\theta) = f(e).$$

Tools used in the proof:

- Maximal torus T
- Weyl integration formula
- Character formula for $\Theta_\pi(t_\theta)$
- Fourier series expansion to express F_f^T in terms of $\Theta_\pi(f)$ $\rightsquigarrow$ need $F_f^T \in C^\infty(T)$
- formula recovering $f(e)$ from the orbital integral $F_f^T(t)$

Cartan subgroups

Example $G = \mathrm{SL}(2,\mathbb{R})$: every element is conjugate to either

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \pm a_t = \pm\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{or} \quad \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$\rightsquigarrow$ two non-conjugate Cartan subgroups T and $A \cup (-A)$
$\rightsquigarrow$ $T^G \cup (\pm A)^G = \{ghg^{-1} : g \in G, h \in T \cup (\pm A)\}$ is dense in G with complement of measure 0

General structure theory

- G linear connected reductive with Cartan involution θ
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresponding Cartan decomposition, $K \subseteq G$ corresponding maximal compact subgroup
- There exist only finitely many non-conjugate θ-stable Cartan subalgebras (i.e. maximal θ-stable abelian subalgebras) $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ of $\mathfrak{g}$ $\quad (\rightsquigarrow \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}))$
  Note: all $\mathfrak{h}_{j,\mathbb{C}}$ are conjugate in $\mathfrak{g}_\mathbb{C}$, in particular: $\mathrm{rank}(G) := \dim \mathfrak{h}_j$ independent of j
- The corresponding Cartan subgroups $H_j = Z_G(\mathfrak{h}_j)$ are abelian, $H_j = (H_j \cap K)(H_j \cap \exp(\mathfrak{p}))$ and the union $H^G_{1,\mathrm{reg}} \cup \ldots \cup H^G_{r,\mathrm{reg}}$ is open and dense in G.
- There exists precisely one $\mathfrak{h}_j$ for which $\mathfrak{h}_j \cap \mathfrak{k}$ resp. $\mathfrak{h}_j \cap \mathfrak{p}$ is of maximal dimension.

Cartan subgroups for $\mathrm{SL}(n,\mathbb{R})$

Example $G = \mathrm{SL}(n,\mathbb{R})$ with $K = \mathrm{SO}(n)$:For $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$ let

$$\mathfrak{h}_j = \left\{ \begin{pmatrix} \begin{pmatrix} t_1 & \theta_1 \\ -\theta_1 & t_1 \end{pmatrix} & & & & & \\ & \ddots & & & & \\ & & \begin{pmatrix} t_j & \theta_j \\ -\theta_j & t_j \end{pmatrix} & & & \\ & & & s_{2j+1} & & \\ & & & & \ddots & \\ & & & & & s_n \end{pmatrix} : \begin{array}{c} t_i, s_i, \theta_i \in \mathbb{R} \\ 2\sum t_i + \sum s_i = 0 \end{array} \right\}$$

$\rightsquigarrow$ non-conjugate θ-stable Cartan subalgebras $\mathfrak{h}_0, \ldots, \mathfrak{h}_{\lfloor \frac{n}{2} \rfloor}$

- $\mathfrak{h}_0 \cap \mathfrak{p} = \mathfrak{h}_0$ of maximal dimension
- $\mathfrak{h}_{\lfloor \frac{n}{2} \rfloor} \cap \mathfrak{k}$ of maximal dimension; moreover: $\mathfrak{h}_{\lfloor \frac{n}{2} \rfloor} \subseteq \mathfrak{k} \Leftrightarrow n = 2$

Weyl Integration Formula

$\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ maximal set of non-conjugate θ-stable Cartan subalgebras, $H_j = Z_G(\mathfrak{h}_j)$ the corresponding Cartan subgroups.After suitable normalization of measures:

Weyl Integration Formula

$$\int_G \varphi(x)\,dx = \sum_{j=1}^{r} \frac{1}{|W(G,H_j)|} \int_{H_j} \int_{G/H_j} \varphi(ghg^{-1})\,d(gH_j)|D_{H_j}(h)|^2\,dh,$$

where

- $W(G,H_j) = N_G(H_j)/Z_G(H_j)$ is the corresponding (finite) Weyl group,
- $D_{H_j}(h)$ the Weyl denominator (expressed in terms of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{j,\mathbb{C}})$).

Weyl Integral Formula – examples

$G = \mathrm{SL}(2,\mathbb{R})$

There are two conjugacy classes of Cartan subgroups:

$$T = K = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \quad \text{and} \quad H = \left\{ \pm a_t = \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

We have $W(G,T) = \{[e]\}$ and $W(G,H) = \{[e],[w_0]\}$ with $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and hence

$$\int_G \varphi(x)\,dx = \int_T \int_{G/T} \varphi(gtg^{-1})|D_T(t)|^2\,d(gT)\,dt + \frac{1}{2}\int_H \int_{G/H} \varphi(ghg^{-1})|D_H(h)|^2\,d(gH)\,dh,$$

where

$$D_T(k_\theta) = e^{i\theta} - e^{-i\theta} = 2i\sin\theta, \qquad D_H(\pm a_t) = \pm(e^t - e^{-t}) = \pm 2\sinh t.$$

Weyl Integral Formula – examples

$G = \mathrm{SL}(2, \mathbb{C})$

There is only one conjugacy class of Cartan subgroups:

$$H = TA = \left\{ t_\theta a_t = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{t+i\theta} & 0 \\ 0 & e^{-t-i\theta} \end{pmatrix} : t, \theta \in \mathbb{R} \right\}.$$

We have $W(G,H) = \{[e], [w_0]\}$ with $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and hence

$$\int_G \varphi(x)\, dx = \frac{1}{2} \int_H \int_{G/H} \varphi(ghg^{-1}) |D_H(h)|^2 \, d(gH)\, dh,$$

where

$$D_H(t_\theta a_t) = 2(\cosh 2t - \cos 2\theta).$$

Weyl Integral Formula – application to characters

Applying the Weyl Integral Formula to $\Theta_\pi(f)=\int_G \Theta_\pi(x)f(x)\,dx$:

$$\Theta_\pi(f)=\sum_{j=1}^r \frac{1}{|W(G,H_j)|}\int_{H_j}\int_{G/H_j}\Theta_\pi(ghg^{-1})f(ghg^{-1})|D_{H_j}(h)|^2\,d(gH)\,dh$$
$$=\sum_{j=1}^r \frac{1}{|W(G,H_j)|}\int_{H_j}\varepsilon_{H_j}(h)\overline{D_{H_j}(h)}\Theta_\pi(h)\times\underbrace{\varepsilon_{H_j}(h)D_{H_j}(h)\int_{G/H_j}f(ghg^{-1})\,d(gH)}_{F_f^{H_j}(h):=}\,dh$$

⇝ orbital integral $F_f^H(h)$ for every Cartan subgroup H
⇝ compute $\Theta_\pi(f)$ for *enough* representations π to recover $F_f^H(h)$ from $\Theta_\pi(f)$
⇝ express $f(e)$ in terms of $F_f^H(h)$ for some H

The case $\mathrm{SL}(2,\mathbb{C})$ – representations

What are the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{C})$?
Consider the minimal parabolic subgroup $P = MAN$ with

$$M = \left\{ t_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\} \qquad A = \left\{ a_t = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}, \qquad N = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}$$

and form the principal series representations

$$\pi_{n,\lambda} = \mathrm{Ind}_P^G(\sigma_n \otimes e^\lambda \otimes 1) \qquad (n \in \mathbb{Z}, \lambda \in \mathbb{C}),$$

with $\sigma_n(t_\theta) = e^{in\theta}$ and $e^\lambda(a_t) = e^{\lambda t}$.

The unitary dual of $G = \mathrm{SL}(2,\mathbb{C})$

- The trivial representation,
- The unitary principal series $\pi_{n,i\lambda}$ $(n \in \mathbb{Z},\ \lambda \in \mathbb{R})$ with $\pi_{n,i\lambda} \simeq \pi_{-n,-i\lambda}$,
- The complementary series $\pi_{0,\lambda}$ $(\lambda \in (-1,1) \setminus \{0\})$ with $\pi_{0,\lambda} \simeq \pi_{0,-\lambda}$.

Note: The Cartan subgroup H splits into $H = TA$ with $T = M$.
$\rightsquigarrow$ $P = MAN$ is *associated to* H

The case $\mathrm{SL}(2,\mathbb{C})$ – characters

The character $\Theta_{n,i\lambda} = \Theta_{\pi_{n,i\lambda}}$ of the induced representation can be expressed in terms of the induction parameters σ_n and e^{λ}, and together with the Weyl Integral Formula we obtain (assuming $f(kgk^{-1}) = f(g)$ for all $k \in K = \mathrm{SU}(2)$):

$$\Theta_{n,i\lambda}(f) = \int_H (\sigma_n \otimes e^{i\lambda})(h) F_f^H(h)\, dh = \int_0^{2\pi} \int_{\mathbb{R}} e^{in\theta} e^{i\lambda t} F_f^H(t_\theta a_t)\, dt\, \frac{d\theta}{2\pi}.$$

One can show that $F_f^H \in C_c^\infty(H)$, so Fourier inversion on $M \simeq \mathbb{T}$ and $A \simeq \mathbb{R}$ yields:

$$F_f^H(t_\theta a_t) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \Theta_{n,i\lambda}(f) e^{-in\theta} e^{-i\lambda t}\, d\lambda.$$

Next: Recover $f(e)$ from $F_f^H(h)$

The case $\mathrm{SL}(2,\mathbb{C})$ – inversion formula

Write $\Sigma(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})=\{\pm\alpha,\pm\overline{\alpha}\}$ for the root system of $\mathfrak{g}_\mathbb{C}=\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ and denote by $\partial(\alpha)$ resp. $\partial(\overline{\alpha})$ the derivative in the direction of α resp. $\overline{\alpha}\in\mathfrak{t}^*\simeq\mathfrak{t}\simeq T$.

Lemma

$$-\frac{1}{2}\partial(\overline{\alpha})\partial(\alpha)F_f^H(e)=(2\pi)^2\cdot f(e)\qquad (f\in C_c^\infty(G)).$$

Proof: Transfer the statement to the Lie algebra (version of the Weyl Integral Formula on $\mathfrak{g}$) and use classical Fourier analysis. □

In the coordinates $(\theta,t)\mapsto t_\theta a_t\in H$ we have $\partial(\overline{\alpha})\partial(\alpha)=\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial t^2}$, hence:

$$F_f^H(t_\theta a_t)=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_\mathbb{R}\Theta_{n,i\lambda}(f)e^{-in\theta}e^{-i\lambda t}\,d\lambda$$

$$\Rightarrow\qquad (2\pi)^3f(e)=\frac{1}{2}\sum_{n\in\mathbb{Z}}\int_\mathbb{R}\Theta_{n,i\lambda}(f)(n^2+\lambda^2)\,d\lambda.$$

The case $\mathrm{SL}(2,\mathbb{C})$ – Plancherel formula

$$(2\pi)^3 f(e) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \Theta_{n,i\lambda}(f)(n^2+\lambda^2)\,d\lambda.$$

Taking into account the symmetry $\pi_{-n,-i\lambda} \simeq \pi_{n,i\lambda} \Rightarrow \Theta_{n,i\lambda}(f) = \Theta_{-n,-i\lambda}(f)$we find:

Plancherel Theorem for $G = \mathrm{SL}(2,\mathbb{C})$

For every $f \in C_c^\infty(G)$:

$$\|f\|^2_{L^2(G)} = \frac{1}{(2\pi)^3}\sum_{n\in\mathbb{Z}}\int_0^\infty \|\Theta_{n,i\lambda}\|^2(n^2+\lambda^2)\,d\lambda.$$

In particular,

$$L^2(G) \simeq \bigoplus_{n\in\mathbb{Z}}\int_0^\infty \pi_{n,i\lambda}\otimes\pi^*_{n,i\lambda}\,d\lambda.$$

Note: The complementary series (which forms a non-empty open subset of $\widehat{G}$) does not contribute to the Plancherel formula.

The case $\mathrm{SL}(2,\mathbb{R})$ – principal series characters

Consider the minimal parabolic subgroup $P = MAN$ with

$$M = \{\pm I\}, \qquad A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}, \qquad N = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix},$$

and form the principal series representations

$$\pi_{\pm,\lambda} = \mathrm{Ind}_P^G(\sigma_\pm \otimes e^\lambda \otimes 1) \qquad (\lambda \in \mathbb{C}),$$

with $\sigma_\pm(-I) = \pm 1$ and $e^\lambda(a_t) = e^{\lambda t}$.

The character $\Theta_{\sigma,i\lambda} = \Theta_{\pi_{\sigma,i\lambda}}$ vanishes on the Cartan subgroup $T = K$, and on H it can be expressed in terms of $\sigma_\pm$ and $e^{i\lambda}$, so together with the Weyl Integral Formula:

$$\rightsquigarrow \quad \Theta_{\pm,i\lambda}(f) = \frac{1}{2} \int_{\mathbb{R}} \left(F_f^H(a_t) \mp F_f^H(-a_t) \right) e^{i\lambda t}\, dt.$$

Similar as for $\mathrm{SL}(2,\mathbb{C})$, we have $F_f^H \in C_c^\infty(H)$ and Fourier inversion yields:

$$\Rightarrow F_f^H(\pm a_t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\Theta_{-,i\lambda}(f) \pm \Theta_{+,i\lambda}(f) \right) e^{-i\lambda t}\, d\lambda.$$

But: $\left.\frac{d}{dt}\right|_{t=0} F_f^H(a_t) = 0$ $\quad\rightsquigarrow$ need more $\Theta_\pi(f)$ to recover $f(e)$

The case $\mathrm{SL}(2,\mathbb{R})$ – discrete series

Definition (discrete series)

An irreducible unitary representation π of G is called *discrete series* if the matrix coefficient $m_{v,w}$ given by

$$m_{v,w}(g) = \langle v, \pi(g)w \rangle \qquad (g \in G)$$

belongs to $L^2(G)$ for all $v, w \in \mathcal{H}_\pi$.

$\rightsquigarrow$ $\mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \hookrightarrow L^2(G)$, $v \otimes w \to m_{v,w}$ is a $G \times G$-equivariant embedding
$\rightsquigarrow$ $\pi \otimes \pi^*$ occurs discretely in the Plancherel formula

Discrete series representations for $\mathrm{SL}(2,\mathbb{R})$: $\pi_n^\pm$ with $n \geq 2$

- subrepresentation of $\pi_{\sigma,\lambda}$ with $(\sigma,\lambda) = ((-1)^n, n-1)$,
- lowest K-type $\sigma_{\pm n}(k_\theta) = e^{\pm in\theta}$.

$\rightsquigarrow$ parameterized by rep's of K

$\rightsquigarrow$ character formula for $\pi_n^\pm$ using the embedding into $\pi_{\sigma,\lambda}$

The case $SL(2,\mathbb{R})$ – discrete series characters

More convenient to treat $\pi_n = \pi_n^+ \oplus \pi_n^-$ and its character Θ_n:

$$\Theta_{n+1}(f) = \frac{1}{2\pi}\int_0^{2\pi}(e^{in\theta} - e^{-in\theta})F_f^T(k_\theta)\,d\theta + \frac{1}{4}\int_{\mathbb{R}}(e^{nt}(1-\operatorname{sgn}t) + e^{-nt}(1+\operatorname{sgn}t))(F_f^H(a_t) - (-1)^{n+1}F_f^H(-a_t))\,dt.$$

In contrast to $F_f^H(\pm a_t)$, the orbital integral $F_f^T(k_\theta)$ has singularities at $\theta = 0$ and $\theta = \pi$:

$$F_f^T(k_{0+}) - F_f^T(k_{0-}) = i\pi F_f^A(a_0) \qquad \text{and} \qquad F_f^T(k_{\pi+}) - F_f^T(k_{\pi-}) = i\pi F_f^A(-a_0).$$

Lemma

$$\lim_{\theta\to 0}\frac{d}{d\theta}F_f^T(k_\theta) = -2\pi i f(e).$$

To involve $\frac{d}{d\theta}F_f^T(k_\theta)$ in $\Theta_{n+1}(f)$, we multiply by n, rewrite $n(e^{in\theta} - e^{-in\theta}) = \frac{1}{i}\frac{d}{d\theta}(e^{in\theta} + e^{-in\theta})$ and integrate by parts.

The case $SL(2,\mathbb{R})$ – Plancherel formula

$$\Rightarrow \sum_{n=1}^{\infty} n\Theta_{n+1}(f) = -\frac{1}{2\pi i}\sum_{k\in\mathbb{Z}}\int_0^{2\pi} e^{ik\theta}\frac{d}{d\theta}F_f^T(k_\theta)\,d\theta + \frac{1}{2\pi i}\int_0^{2\pi}\frac{d}{d\theta}F_f^T(k_\theta)\,d\theta$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{d}{dt}\left(F_f^H(a_t)+(-1)^n F_f^H(-a_t)\right)\,dt.$$

- First term (Fourier series + regularity of $\frac{d}{d\theta}F_f^T(k_\theta)$) $\rightsquigarrow f(e)$
- Second term (Behaviour of $F_f^T(k_\theta)$ at $\theta = 0,\pi$) $\rightsquigarrow F_f^H(\pm a_0) \rightsquigarrow \int_{\mathbb{R}}\Theta_{-,i\lambda}(f)\,d\lambda$
- Third term (Parseval's Formula + Fourier transform of $F_f^H(a_t)$) $\rightsquigarrow \int_{\mathbb{R}}\frac{\lambda^2}{n^2+\lambda^2}\Theta_{\pm,i\lambda}\,d\lambda$

$$\Rightarrow \quad 2\pi f(e) = \sum_{n=1}^{\infty} n\Theta_{n+1}(f) + \frac{1}{4}\int_{\mathbb{R}}\Theta_{+,i\lambda}(f)\lambda\tanh(\tfrac{\pi\lambda}{2})\,d\lambda + \frac{1}{4}\int_{\mathbb{R}}\Theta_{-,i\lambda}(f)\lambda\coth(\tfrac{\pi\lambda}{2})\,d\lambda.$$

Plancherel Theorem for $SL(2,\mathbb{R})$

$$L^2(G) \simeq \bigoplus_{n=2}^{\infty}\left([\pi_n^+\otimes(\pi_n^+)^*]\oplus[\pi_n^+\otimes(\pi_n^+)^*]\right)\oplus\bigoplus_{\pm}\int_0^{\infty}[\pi_{\pm,i\lambda}\otimes\pi_{\pm,i\lambda}^*]\,d\lambda.$$

The case $\mathrm{SL}(2,\mathbb{R})$ – summary

- Two conjugacy classes of Cartan subgroups:
  - $H = MA \rightsquigarrow$ minimal parabolic subgroup $P = MAN \rightsquigarrow$ unitary principal series $\pi_{\sigma,i\lambda}$
  - $T = K \rightsquigarrow$ discrete series $\pi_n^\pm$
- Weyl integration formula + character formulas:

$$\Theta_n(f) = \int_T (\dots) F_f^T(t)\,dt + \int_H (\dots) F_f^H(h)\,dh$$
$$\Theta_{\pm,i\lambda}(f) = \int_H (\dots) F_f^H(h)\,dh$$

- Using Euclidean Fourier analysis on H and Fourier series on T + singularities of F_f^T $\rightsquigarrow$ solve for $F_f^T(k_\theta)$ (or rather $\frac{d}{d\theta}\Big|_{\theta=0} F_f^T(k_\theta)$)

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