Representations of reductive groups

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Outline

What are these talks about?

Langlands classification: big picture

Introduction to Harish-Chandra modules

$(\mathfrak{h}, L)$-modules as ring modules

Langlands classification: some details

Cartan subgroups of real reductive groups

Langlands classification: getting explicit

Representations of $K(\mathbb{R})$

What real reductive groups?

Old days: assumed $G(\mathbb{R})$ connected semisimple.

Problem is that $G(\mathbb{R})$ is studied using Levi subgroups; these aren't connected even if G is.

Here are some possible assumptions for us:

1. Narrowest: G complex connected reductive algebraic defined over $\mathbb{R}$, $G(\mathbb{R})$ = real points.
2. Somewhat weaker: $G(\mathbb{R})$ is transpose-stable subgp of $GL(n, \mathbb{R})$ with $G(\mathbb{R})/G(\mathbb{R})_0$ finite.
3. Still weaker: $G(\mathbb{R})$ is finite cover of a group as in (2).

General notation: $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G(\mathbb{R}))$, $\mathfrak{g} = \mathfrak{g}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

Everything I say holds exactly under (1);

lots is still true under the (strictly weaker) (2);

most things work under (3).

Structure of $G(\mathbb{R})$

$$G(\mathbb{R}) \hookrightarrow GL(n,\mathbb{R}), \quad \text{stable by transpose}, \quad G(\mathbb{R})/G(\mathbb{R})_0 \text{ finite.}$$

Cartan involution of $GL(n,\mathbb{R})$ is automorphism $\theta(g) = {}^t g^{-1}$.

Recall *polar decomposition*:

$$GL(n,\mathbb{R}) = O(n) \times \exp(\text{symmetric matrices}).$$

$$= GL(n,\mathbb{R})^{\theta} \times \exp(\mathfrak{gl}(n,\mathbb{R})^{-\theta})$$

Inherited by $G(\mathbb{R})$ as Cartan decomposition for $G(\mathbb{R})$:

$$K(\mathbb{R}) = G(\mathbb{R})^{\theta} = O(n) \cap G(\mathbb{R}),$$

$$\mathfrak{s}(\mathbb{R}) = \mathfrak{g}(\mathbb{R})^{-\theta} = \text{symm matrices in } \mathfrak{g}(\mathbb{R})$$

$$S(\mathbb{R}) = \exp(\mathfrak{s}(\mathbb{R})) = \text{pos def symm matrices in } G(\mathbb{R}),$$

$$G(\mathbb{R}) = K(\mathbb{R}) \times S(\mathbb{R}) \simeq K(\mathbb{R}) \times \mathfrak{s}(\mathbb{R}).$$

Nice structures on $G(\mathbb{R})$ come from nice structures on $K(\mathbb{R})$ by solving differential equations along $S(\mathbb{R})$.

What representations (A)?

As good analytic C^*-algebra people, you understand

Definition. Unitary representation of $G(\mathbb{R})$ on Hilbert space $\mathcal{H}_\pi$ is weakly continuous homomorphism

$$\pi\colon G \to U(\mathcal{H}_\pi).$$

Irreducible if $\mathcal{H}_\pi$ has exactly two closed $G(\mathbb{R})$-invt subspaces.

Chevalley told Harish-Chandra to weaken this definition.

Definition. Representation of reductive $G(\mathbb{R})$ on loc cvx complete V_π is weakly continuous group homomorphism

$$\pi\colon G \to GL(V_\pi)$$

Get a new loc cvx complete $V_\pi^\infty \subset V_\pi$ on which π^∞ differentiates to action of $U(\mathfrak{g})$.

Define $\mathfrak{Z}(\mathfrak{g}) = U(\mathfrak{g})^{\mathrm{Ad}(G(\mathbb{R}))}$. Schur's lemma suggests that $\mathfrak{Z}(\mathfrak{g})$ should act by scalars on V_π^∞ for irreducible π.

Always true for π unitary (Segal), fails sometimes for nonunitary π on any noncompact $G(\mathbb{R})$ (Soergel).

What representations (B)?

Definition (Harish-Chandra) Rep π of $G(\mathbb{R})$ on complete loc cvx V_π is quasisimple if $\mathfrak{Z}(\mathfrak{g})$ acts by scalars on V_π^∞.

You know to care about $\widehat{G(\mathbb{R})}_u$ = unitary equivalence classes of irr unitary representations.

HC says to care about larger $\widehat{G(\mathbb{R})}$ = infinitesimal equivalence classes of irr quasisimple π.

Defining infinitesimal equivalence is a bit complicated; soon...

To see the value of this, helpful to introduce $\widehat{G(\mathbb{R})}_h$ = infl equiv classes of irr quasisimple π with nonzero (maybe indefinite) invariant Hermitian form.

$$\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})}.$$

You know that the left term is interesting. I claim that it's best understood by understanding the right term and the two inclusions. . .

What representations (C)?

$\widehat{G(\mathbb{R})}_u$	$\subset$	$\widehat{G(\mathbb{R})}_h$	$\subset$	$\widehat{G(\mathbb{R})}$
unitary	$\subset$	hermitian	$\subset$	quasisimple
desirable	$\subset$	acceptable	$\subset$	available

Langlands classification beautifully describes $\widehat{G(\mathbb{R})}$ as complex algebraic variety.

Knapp-Zuckerman describe $\widehat{G(\mathbb{R})}_h$ as real points of this alg variety: fixed points of simple complex conjugation.

$\widehat{G(\mathbb{R})}_u$ is cut out inside $\widehat{G(\mathbb{R})}_h$ by real algebraic inequalities, more or less computed by Adams, van Leeuwen, Trapa, V.

What can we ask about representations?

Start with a reasonable category of representations...
Example: cplx reductive $\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$; BGG category $\mathcal{O}$ consists of $U(\mathfrak{g})$-modules V subject to

1. fin gen: $\exists V_0 \subset V, \ \dim V_0 < \infty, \ U(\mathfrak{g})V_0 = V$.
2. $\mathfrak{b}$-locally finite: $\forall v \in V, \ \dim U(\mathfrak{b})v < \infty$.
3. $\mathfrak{h}$-semisimple: $V = \sum_{\gamma \in \mathfrak{h}^*} V_\gamma$.

Want precise information about reps in the category.
Example: V in category $\mathcal{O}$

1. $\dim V_\gamma$ is *almost polynomial* as function of γ.
2. V has a *formal character* $\left[\sum_{\lambda \in \mathfrak{h}^*} a_V(\lambda) e^\lambda\right] / \Delta$.

Want construction/classification of reps in the category.
Example: $\lambda \in \mathfrak{h}^* \rightsquigarrow I(\lambda) =_{\text{def}} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda =$ *Verma module*.

1. (STRUCTURE THM): $I(\lambda)$ has highest weight $\mathbb{C}_\lambda \hookrightarrow I(\lambda)^{\mathfrak{n}}$.
2. (QUOTIENT THM): $I(\lambda)$ has unique irr quo $J(\lambda)$.
3. (CLASSIF THM): Each irr in $\mathcal{O}$ is $J(\lambda)$, unique $\lambda \in \mathfrak{h}^*$.

How do you do that?

$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ roots, Δ^+ roots in $\mathfrak{n}$.

$\rightsquigarrow$ partial order on $\mathfrak{h}^*$:

$$\mu' \le \mu \iff \mu' \in \mu - \mathbb{N}\Delta^+$$
$$\iff \mu' = \mu - \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \quad (n_\alpha \in \mathbb{N})$$

Proposition. Suppose $V \in \mathcal{O}$.

1. If $V \ne 0, \exists$ *maximal* $\mu \in \mathfrak{h}^*$ subject to $V_\mu \ne 0$.
2. If $\mu \in \mathfrak{h}^*$ is maxl subj to $V_\mu \ne 0$, then $V_\mu \subset V^{\mathfrak{n}}$.
3. If $V \ne 0, \exists \mu$ with $0 \ne V_\mu \subset V^{\mathfrak{n}}$.
4. $\forall \lambda \in \mathfrak{h}^*$, $\mathrm{Hom}_{\mathfrak{g}}(I(\lambda), V) \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda, V^{\mathfrak{n}})$.

Parts (1)–(3) guarantee existence of "highest weights;" based on formal calculations with lattices in vector spaces, and $\mathfrak{n} \cdot V_{\mu'} \subset \sum_{\alpha \in \Delta^+} V_{\mu'+\alpha}$.

Sketch of proof of (4):

$$\mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, V) \simeq \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, V) = \mathrm{Hom}_{U(\mathfrak{h})}(\mathbb{C}_\lambda, V^{\mathfrak{n}}).$$

First isom: "change of rings." Second: $\mathfrak{n} \cdot \mathbb{C}_\lambda =_{\mathrm{def}} 0$.

Moral of the story

For category $\mathcal{O}$, three key ingredients:

1. Change of rings $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \cdot \rightsquigarrow$ Verma mods $I(\lambda)$.
2. Universality: $\mathrm{Hom}_{\mathfrak{g}}(I(\lambda), V) \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda, V^{\mathfrak{n}})$.
3. Highest weight exists: J irr $\implies J^{\mathfrak{n}} \neq 0$.

#2 is homological alg, #3 is comb/geom in $\mathfrak{h}^*$.

Irrs J in $\mathcal{O} \leftrightsquigarrow \lambda \in \mathfrak{h}^*$ characterized by $\mathbb{C}_\lambda \subset J(\lambda)^{\mathfrak{n}}$.

Same three ideas apply to $G(\mathbb{R})$ representations.

Technical problem: change of rings isn't projective, so $\otimes \rightsquigarrow$ Tor.

Parallel problem: $J^{\mathfrak{n}} = H^0(\mathfrak{n}, J) \rightsquigarrow$ derived functors $H^p(\mathfrak{n}, J)$.

Conclusion will be: irr $G(\mathbb{R})$-reps $J \leftrightsquigarrow \gamma \in \widehat{H(\mathbb{R})}$, some Cartan $H(\mathbb{R}) \subset G(\mathbb{R})$; char by $\mathbb{C}_\gamma \subset H^s(\mathfrak{n}, J)$.

Next topic: Harish-Chandra's algebraization of rep theory, making possible the program outlined above.

Principal series for $SL(2,\mathbb{R})$ (skip this!)

To understand how Harish-Chandra studied reductive group representations, need a serious example.

But there isn't time; so look at these slides on your own!

Use principal series repns for $SL(2,\mathbb{R}) =_{\text{def}} G(\mathbb{R})$.

$G(\mathbb{R}) \curvearrowright \mathbb{R}^2$, so get rep of $G(\mathbb{R})$ on functions on $\mathbb{R}^2$:

$$[\rho(g)f](v) = f(g^{-1}\cdot v).$$

Lie algs easier than Lie gps $\rightsquigarrow$ write $\mathfrak{sl}(2,\mathbb{R})$ action, basis

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Action on functions on $\mathbb{R}^2$ is by vector fields:

$$\rho(D)f = -x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2}, \quad \rho(E) = -x_2\frac{\partial f}{\partial x_1}, \quad \rho(F) = -x_1\frac{\partial f}{\partial x_2}.$$

General principle: representations on function spaces are reducible $\leftrightsquigarrow$ exist $G(\mathbb{R})$-invt differential operators.

Euler deg operator $E = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ commutes with $G(\mathbb{R})$.

Conclusion: interesting reps of $G(\mathbb{R})$ on eigenspaces of E.

Principal series for $SL(2,\mathbb{R})$ (also skip)

Previous slide: expect interesting reps of $G(\mathbb{R}) = SL(2,\mathbb{R})$ on homogeneous functions on $\mathbb{R}^2$.

For $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, define

$$W^{\nu,\epsilon} = \{f\colon (\mathbb{R}^2 - 0) \to \mathbb{C} \mid f(tx) = |t|^{-\nu-1}\operatorname{sgn}(t)^{\epsilon} f(x)\},$$

functions on the plane homog of degree $-(\nu+1,\epsilon)$.

$\nu \rightsquigarrow \nu+1$ simplifies MANY things later...

Study $W^{\nu,\epsilon}$ by restriction to circle $\{(\cos\theta,\sin\theta)\}$:

$$W^{\nu,\epsilon} \simeq \{w\colon S^1 \to \mathbb{C} \mid w(-s) = (-1)^{\epsilon} w(s)\},\ f(r,\theta) = r^{-\nu-1} w(\theta).$$

Compute Lie algebra action in polar coords using

$$\frac{\partial}{\partial x_1} = -x_2\frac{\partial}{\partial\theta} + x_1\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial x_2} = x_1\frac{\partial}{\partial\theta} + x_2\frac{\partial}{\partial r},$$

$$\frac{\partial}{\partial r} = -\nu - 1, \qquad x_1 = \cos\theta, \qquad x_2 = \sin\theta.$$

Plug into formulas on preceding slide: get

$$\rho^{\nu,\epsilon}(D) = 2\sin\theta\cos\theta\frac{\partial}{\partial\theta} + (-\cos^2\theta + \sin^2\theta)(\nu+1),$$

$$\rho^{\nu,\epsilon}(E) = \sin^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1),$$

$$\rho^{\nu,\epsilon}(F) = -\cos^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1).$$

A more suitable basis (skip this too!)

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2,\mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^{\nu,\epsilon}(D) = 2\sin\theta\cos\theta\frac{\partial}{\partial\theta} + (-\cos^2\theta + \sin^2\theta)(\nu+1),$$
$$\rho^{\nu,\epsilon}(E) = \sin^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1),$$
$$\rho^{\nu,\epsilon}(F) = -\cos^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1).$$

Hard to make sense of. Clear: family of reps analytic (actually linear) in complex parameter ν.

Big idea: see how properties change as function of ν.

Problem: $\{D, E, F\}$ adapted to wt vectors for diagonal Cartan subalgebra; rep $\rho^{\nu,\epsilon}$ has no such wt vectors.

But rotation matrix $E - F$ acts simply by $\partial/\partial\theta$.

Suggests new basis of the complexified Lie algebra:

$$H = -i(E-F), \quad X = \frac{1}{2}(D+iE+iF), \quad Y = \frac{1}{2}(D-iE-iF).$$

$$\rho^{\nu,\epsilon}(H) = \frac{1}{i}\frac{\partial}{\partial\theta}, \ \rho^{\nu,\epsilon}(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right), \ \rho^{\nu,\epsilon}(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right).$$

Principal series, bad news (not for us!)

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2,\mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^{\nu,\epsilon}(H) = \frac{1}{i}\frac{\partial}{\partial\theta}, \quad \rho^{\nu,\epsilon}(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right), \quad \rho^{\nu,\epsilon}(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right).$$

These ops act simply on basis $w_m(\cos\theta, \sin\theta) = e^{im\theta}$:

$$\begin{aligned} \rho^{\nu,\epsilon}(H)w_m &= mw_m, \\ \rho^{\nu,\epsilon}(X)w_m &= \frac{1}{2}(m+\nu+1)w_{m+2}, \\ \rho^{\nu,\epsilon}(Y)w_m &= \frac{1}{2}(-m+\nu+1)w_{m-2}. \end{aligned}$$

Suggests reasonable function space to consider:

$$\begin{aligned} W^{\nu,\epsilon,K(\mathbb{R})} &= \text{fns homog of deg } (\nu,\epsilon), \text{ finite under rotation} \\ &= \text{span}(\{w_m \mid m \equiv \epsilon \pmod 2\}). \end{aligned}$$

$W^{\nu,\epsilon,K(\mathbb{R})}$ has beautiful rep of $\mathfrak{g}$: irr for most ν, easy submods otherwise. Not preserved by $G(\mathbb{R}) = SL(2,\mathbb{R})$: $\exp(A) \in G(\mathbb{R}) \rightsquigarrow \sum A^k/k!$: $A^k \curvearrowright W^{\nu,\epsilon,K(\mathbb{R})}$, sum not.

Principal series: good news (last skip!)

Original question was action of $G(\mathbb{R}) = SL(2, \mathbb{R})$ on

$$W^{\nu,\epsilon,\infty} = \{f \in C^\infty(\mathbb{R}^2 - 0) \mid f \text{ homog of deg } -(\nu + 1, \epsilon)\} :$$

what are the closed $G(\mathbb{R})$-invt subspaces. . . ?

Found nice subspace $W^{\nu,\epsilon,K(\mathbb{R})}$, explicit basis, explicit action of Lie algebra $\rightsquigarrow$ easy to describe $\mathfrak{g}$–invt subspaces.

Theorem (Harish-Chandra) There is one-to-one corr

$$\text{closed } G(\mathbb{R})\text{-invt } S \subset W^{\nu,\epsilon,\infty} \leftrightsquigarrow \mathfrak{g}(\mathbb{R})\text{-invt } S^K \subset W^{\nu,\epsilon,K}$$

$$S \rightsquigarrow K\text{-finite vectors in } S, \quad S^K \rightsquigarrow \overline{S^K}.$$

Content of thm: closure carries $\mathfrak{g}$-invt to G-invt.

Why this isn't obvious: $SO(2)$ acting by translation on $C^\infty(S^1)$. Lie alg acts by $\frac{d}{d\theta}$, so closed subspace

$$E = \{f \in C^\infty(S^1) \mid f(\cos\theta, \sin\theta) = 0, \theta \in (-\pi/2, \pi/2) + 2\pi\mathbb{Z}\}$$

is preserved by $\mathfrak{so}(2)$; *not* preserved by rotation.

Reason: Taylor series for in $f \in E$ doesn't converge to f.

Making representations algebraic

Back to general setting: $G(\mathbb{R})$ real reductive,
$\theta\colon G(\mathbb{R}) \to G(\mathbb{R})$ Cartan involution, $\mathfrak{s}(\mathbb{R}) = \mathfrak{g}(\mathbb{R})^{-\theta}$.

$K(\mathbb{R}) = G(\mathbb{R})^{\theta}$ compact subgroup.

Recall polar decomposition $G(\mathbb{R}) = K(\mathbb{R}) \times \exp(\mathfrak{s}_0)$.

Nice structures on $G(\mathbb{R})$ come from nice structures on $K(\mathbb{R})$ by solving differential equations along S.

(ρ, W) rep on complete loc cvx W; had smaller space

$$W^{\infty} = \{w \in W \mid G(\mathbb{R}) \to W,\ \ g \mapsto \rho(g)w \text{ smooth}\}.$$

Similarly define two more smaller complete loc cvx spaces

$$W^{K(\mathbb{R})} = \{w \in W \mid \dim \operatorname{span}(\rho(K(\mathbb{R}))w) < \infty\},$$

$$W^{K(\mathbb{R}),\infty} = \{w \in W^{\infty} \mid \dim \operatorname{span}(\rho(K(\mathbb{R}))w) < \infty\}$$

Definition. The Harish-Chandra-module of W is $W^{K(\mathbb{R}),\infty}$: representation of Lie algebra $\mathfrak{g}(\mathbb{R})$ and of group $K(\mathbb{R})$.

Easy (two slides below!) to define $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$-modules.

Group reps and Lie algebra reps

$G(\mathbb{R})$ reductive $\supset K(\mathbb{R})$ max cpt, $\mathfrak{Z}(\mathfrak{g}) = U(\mathfrak{g})^{\mathrm{Ad}(G)}$.

Recall (π, V) is *quasisimple* if $\pi^\infty(z) =$ scalar, $z \in \mathfrak{Z}(\mathfrak{g})$.

Theorem (Segal, Harish-Chandra)

1. Any irreducible $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$-module is quasisimple.
2. Any irreducible unitary rep of $G(\mathbb{R})$ is quasisimple.
3. Suppose V quasisimple rep of $G(\mathbb{R})$. Then $W \mapsto W^{K(\mathbb{R}),\infty}$ is bijection between subrepresentations
$$(\text{closed } W \subset V) \leftrightarrow (W^{K(\mathbb{R}),\infty} \subset V^{K(\mathbb{R}),\infty}).$$
4. (irreducible quasisimple reps of $G(\mathbb{R})$) $\rightsquigarrow$ (irreducible $(\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))$-modules), $W_\pi \rightsquigarrow W_\pi^{K(\mathbb{R}),\infty}$ is surjective.

Idea of proof: $G(\mathbb{R})/K(\mathbb{R}) \simeq \mathfrak{s}_0$, vector space. Describe anything analytic on $G(\mathbb{R})$ by Taylor expansion along $K(\mathbb{R})$.

Category of $(\mathfrak{h}(\mathbb{R}), L(\mathbb{R}))$-modules

Setting: $\mathfrak{h}(\mathbb{R}) \supset \mathfrak{l}(\mathbb{R})$ real Lie algebras, $L(\mathbb{R})$ compact Lie group acting on $\mathfrak{h}(\mathbb{R})$ by Lie algebra automorphisms Ad.

Definition. An $(\mathfrak{h}(\mathbb{R}), L(\mathbb{R}))$-module is complex vector space W, with reps of $\mathfrak{h}(\mathbb{R})$ and of $L(\mathbb{R})$, subject to

1. each $w \in W$ belongs to fin-diml $L(\mathbb{R})$-invt W_0, so that action of $L(\mathbb{R})$ on W_0 continuous (hence smooth);
2. differential of $L(\mathbb{R})$ action is $\mathfrak{l}(\mathbb{R})$ action;
3. $\forall k \in L(\mathbb{R})$, $Z \in \mathfrak{h}(\mathbb{R})$, $w \in W$, $k \cdot (Z \cdot (k^{-1} \cdot w)) = [\mathrm{Ad}(k)(Z)] \cdot w$.

Condition (3) is automatic if $L(\mathbb{R})$ connected.

Write $\mathcal{M}(\mathfrak{h}(\mathbb{R}), L(\mathbb{R}))$ for category of $(\mathfrak{h}(\mathbb{R}), L(\mathbb{R}))$-modules.

Proposition. Taking smooth $K(\mathbb{R})$-fin vecs is functor

$$\text{(reps of } G(\mathbb{R}) \text{ on complete loc cvx } W)$$
$$\longrightarrow (\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))\text{-modules } W^{K(\mathbb{R}),\infty}.$$

But it's easier to use reps of complex Lie algebras...

Complexified Lie algebras

real Lie algebra $\mathfrak{h}(\mathbb{R}) \rightsquigarrow$ complex Lie algebra

$$\mathfrak{h} = \mathfrak{h}(\mathbb{C}) =_{\text{def}} \mathfrak{h}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \{X + iY \mid X, Y \in \mathfrak{h}(\mathbb{R}).\}$$

complexification of $\mathfrak{h}(\mathbb{R})$.

Proposition. Representation (π_0, V) of $\mathfrak{h}(\mathbb{R}) \leftrightsquigarrow$ representation (π_1, V) of $\mathfrak{h}(\mathbb{C})$:

$$\pi_1(X + iY) = \pi_0(X) + i\pi_0(Y), \qquad \pi_0(X) = \pi_1(X).$$

Identification $\pi_0 \leftrightsquigarrow \pi_1$ is perfect; write π for both.

Convenient to express as modules for an algebra:

Proposition. Reps of real Lie alg $\mathfrak{h}(\mathbb{R}) \leftrightsquigarrow$ modules for complex enveloping algebra $U(\mathfrak{h})$.

Seek to extend this to $(\mathfrak{h}(\mathbb{R}), L(\mathbb{R}))$-modules.

Complexified compact Lie groups

Complexification also works for compact groups...

real compact $L(\mathbb{R}) \subset U(n) \rightsquigarrow$ complex reductive alg

$$L = L(\mathbb{C}) =_{\text{def}} L(\mathbb{R}) \exp(i\mathfrak{l}(\mathbb{R}) \subset GL(n, \mathbb{C})$$

complexification of $L(\mathbb{R})$.

Coordinate-free definition:

reg fns on $L(\mathbb{C})$ = $L(\mathbb{R})$-finite $\mathbb{C}$-valued fns on $L(\mathbb{R})$

Proposition. Fin-diml continuous (π_0, V) of $L(\mathbb{R}) \leftrightsquigarrow$ fin-diml algebraic representation (π_1, V) of $L(\mathbb{C})$:

$$\pi_1(l \exp(iY)) = \pi_0(l) \exp(id\pi_0(Y)), \qquad \pi_0(l) = \pi_1(l).$$

Identification $\pi_0 \leftrightsquigarrow \pi_1$ is perfect; write π for both.

$L(\mathbb{R})$-finite cont reps of $L(\mathbb{R})$ = algebraic reps of $L(\mathbb{C})$.

Category of $(\mathfrak{h}, L)$-modules

Now we can complexify Harish-Chandra's category…

Setting: $\mathfrak{h} \supset \mathfrak{l}$ complex Lie algebras, L complex algebraic acting on $\mathfrak{h}$ by Lie algebra automorphisms Ad.

Definition. An $(\mathfrak{h}, L)$-module is complex vector space W, with reps of $\mathfrak{h}$ and of L, subject to

1. L action is algebraic (hence smooth);
2. differential of L action is $\mathfrak{l}$ action;
3. For $k \in L$, $Z \in \mathfrak{h}$, $w \in W$,
   $k \cdot (Z \cdot (k^{-1} \cdot w)) = [\mathrm{Ad}(k)(Z)] \cdot w$.

Write $\mathcal{M}(\mathfrak{h}, L)$ for category of $(\mathfrak{h}, L)$-modules.

Proposition. Taking smooth K-finite vecs is functor

$$W \in (\text{reps of } G(\mathbb{R}) \text{ on complete locally convex space}) \longrightarrow W^{K,\infty} \in \mathcal{M}(\mathfrak{g}, K)$$

.

Representations and R-modules

Rings and modules familiar and powerful $\rightsquigarrow$ try to make representation categories into module categories. Saw

Category of reps of $\mathfrak{h}(\mathbb{R})$ = category of $U(\mathfrak{h})$-modules.

Seek parallel for locally finite reps of compact $L(\mathbb{R})$:

$$R(L) = \text{conv alg of } \mathbb{C}\text{-valued } L\text{-finite msres on } L(\mathbb{R})$$

$$\simeq_{\text{(Peter-Weyl)}} \left[\sum_{(\mu, E_\mu) \in \widehat{L}} \mathrm{End}(E_\mu)\right]$$

$1 \notin R(L)$ if $L(\mathbb{R})$ is infinite: convolution identity is point measure at $e \in L(\mathbb{R})$, not L-finite.

$$\alpha \subset \widehat{L} \text{ finite } \rightsquigarrow 1_\alpha =_{\text{def}} \sum_{\mu \in \alpha} \mathrm{Id}_\mu \in R(L).$$

Elements 1_α are approximate identity: $\forall r \in R(L)\ \ \exists \alpha(r)$ finite so $1_\beta \cdot r = r \cdot 1_\beta = r$ if $\beta \supset \alpha(r)$.

$R(L)$-module M is approximately unital if $\forall m \in M\ \ \exists \alpha(m)$ finite so $1_\beta \cdot m = m$ if $\beta \supset \alpha(m)$.

Alg reps of L = approximately unital $R(L(\mathbb{R}))$-modules.

R-mod $=_{\text{def}}$ category of approximately unital R-modules.

Hecke algebras

Setting: $\mathfrak{h} \supset \mathfrak{l}$ cplx Lie algs, L reductive alg $\curvearrowright \mathfrak{h}$ by Lie alg automorphisms Ad.

Definition. The Hecke algebra $R(\mathfrak{h}, L)$ is

$$R(\mathfrak{h}, L) = U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} R(L)$$
$$\simeq [\text{conv alg of } L\text{-finite } U(\mathfrak{h})\text{-valued msres on } L(\mathbb{R})]/U(\mathfrak{l})$$

$R(\mathfrak{h}, L)$ inherits approx identity from subalgebra $R(L)$.

Proposition. $\mathcal{M}(\mathfrak{h}, L) = R(\mathfrak{h}, L)\text{-mod}$: $(\mathfrak{h}, L)$ modules are approximately unital modules for Hecke algebra $R(\mathfrak{h}, L)$.

Immediate corollary: $\mathcal{M}(\mathfrak{h}, L)$ has projective resolutions, so derived functors. . .

Langlands classification

Theorem (Langlands) Irreducible representations of a real reductive group $G(\mathbb{R})$ are in one-to-one correspondence

$$(H(\mathbb{R}), \gamma)/(G(\mathbb{R}) \text{ conjugacy}) \leftrightsquigarrow J(H(\mathbb{R}), \gamma) \qquad \text{with}$$

1. $H(\mathbb{R}) \subset G(\mathbb{R})$ is a Cartan subgroup, $\gamma \in \widehat{H}(\mathbb{R})$ a character;
2. γ nontrivial on each compact imaginary simple coroot; and
3. γ nontrivial on each simple real coroot.

Equivalently,

$$\widehat{G(\mathbb{R})} = \coprod_{H(\mathbb{R})/G(\mathbb{R})} \widehat{H(\mathbb{R})}_{\text{reg}} / W(G(\mathbb{R}), H(\mathbb{R})).$$

(2) is the “regularity” condition in Langlands classification for K;

(3) excludes the reducible tempered principal series of $SL(2, \mathbb{R})$

$J(H(\mathbb{R}), \gamma)$ characterized by occurrence of $\gamma - \rho$ in $H(\mathbb{R})$ action on $H^s(\mathfrak{n}, J)$ (some Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$).

Remaining lies: omitted translate of γ by ρ, choice of pos imag roots.

Next time: what $H(\mathbb{R})$ and $W(G(\mathbb{R}), H(\mathbb{R}))$ look like.

What have we done?

Harish-Chandra's notion of all irreducible representations π of $G(\mathbb{R})$: continuous irreducible on complete loc cvx top vec space W_π, quasisimple: $U(\mathfrak{g})^{\mathrm{Ad}(G(\mathbb{R}))}$ acts by scalars.

$\rightsquigarrow W_\pi^{K,\infty}$ irr $(\mathfrak{g}, K)$-module of K-finite smooth vecs.

$\widehat{G(\mathbb{R})} =_{\mathrm{def}}$ infinitesimal equiv classes of irr quasisimple, so

$$\widehat{G(\mathbb{R})} \simeq_{\mathrm{def}} \text{simple } R(\mathfrak{g}, K)\text{-modules.}$$

Langlands classification proceeds by category $\mathcal{O}$ strategy:

1. construct (complicated) $R(\mathfrak{g}, K)$-modules from (simple) $R(\mathfrak{h}, H \cap K)$-modules by change-of-rings functors;
2. prove exhaustion using universality properties involving Lie algebra cohomology.

If you've read Langlands, this summary may look absurd. But...

Change-of-rings includes parabolic induction.

Lie algebra cohom can come from asymptotic exp of matrix coeffs.

Feel better?

END OF LECTURE ONE

BEGINNING OF LECTURE TWO

Cartan subgroups

Said that Langlands parametrized irr reps of real reductive $G(\mathbb{R})$ by characters of Cartan subgroups $H(\mathbb{R})$.

To make precise/concrete, need structure of $H(\mathbb{R})$.

Assume (replace $H(\mathbb{R})$ by conjugate) $\theta(H(\mathbb{R})) = H(\mathbb{R})$.

Set $T(\mathbb{R}) = H(\mathbb{R})^\theta = H(\mathbb{R}) \cap K(\mathbb{R})$ compact

Set $\mathfrak{a}_0 = \mathfrak{h}(\mathbb{R})^{-\theta}, \quad A = \exp(\mathfrak{a}_0)$ vector group.

$$H(\mathbb{R}) = T(\mathbb{R}) \times A$$

$$\widehat{H(\mathbb{R})} = (\text{chars of } T(\mathbb{R})) \times (\mathfrak{a}^*)$$
$$= (\text{nearly lattice}) \times (\text{complex vector space}).$$

$\widehat{G(\mathbb{R})} =$ countable union of complex vector spaces.

Examples of Cartan subgroups

$\mathrm{Sp}(2n,\mathbb{R})$ = linear maps of $2n$-dimensional real E preserving nondegenerate skew-symm bilinear form ω.

1st construction: U n-diml real $E = U \oplus U^*$,

$$\omega((u_1,\lambda_1),(u_2,\lambda_2)) = \lambda_1(u_2) - \lambda_2(u_1).$$

Get $GL(U) \hookrightarrow \mathrm{Sp}(E)$, $\quad g \cdot (u,\lambda) = (g \cdot u, {}^t g^{-1} \cdot \lambda)$.

$\rightsquigarrow$ Cartan subgp $H_{n,0,0}(\mathbb{R}) = GL(1,\mathbb{R})^n \subset GL(n,\mathbb{R}) \subset \mathrm{Sp}(2n,\mathbb{R})$.

2nd construction: F n-diml complex with nondeg Herm form μ, $\omega(f_1,f_2) = \mathrm{Im}(\mu(f_1,f_2))$ (on real space $F|_{\mathbb{R}}$).
Get unitary group $U(F) \hookrightarrow \mathrm{Sp}(F|_{\mathbb{R}})$.

$\rightsquigarrow$ Cartan $H_{0,0,n}(\mathbb{R}) = U(1)^n \subset U(p,q) \subset \mathrm{Sp}(2n,\mathbb{R})$.

3rd construction: $n = 2m$ even, V m-diml complex, $\omega_{\mathbb{C}}$ on $F = V \oplus V^*$ as in 1st, $\omega_{\mathbb{R}} = \mathrm{Re}(\omega_{\mathbb{C}})$ on $F|_{\mathbb{R}}$.
Get $\underbrace{GL(V) \hookrightarrow \mathrm{Sp}(F)}_{\text{complex algebraic}} \hookrightarrow \underbrace{\mathrm{Sp}(F|_{\mathbb{R}})}_{\text{real}}$.

$\rightsquigarrow$ Cartan $H_{0,m,0} = GL(1,\mathbb{C})^m \subset GL(m,\mathbb{C}) \subset \mathrm{Sp}(2m,\mathbb{C}) \subset \mathrm{Sp}(4m,\mathbb{R})$.

Any Cartan: $H_{a,b,c} \simeq (\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b \times U(1)^c \quad (n = a + 2b + c)$.

$T_{a,b,c} = \{\pm 1\}^a \times U(1)^{b+c}, \qquad A_{a,b,c} = \mathbb{R}^{a+b}$

Cartans, eigenvalues, Weyl groups/$\mathbb{C}$

$g \in G(\mathbb{C}) = \mathrm{Sp}(2n, \mathbb{C})$ (complex reductive) has $2n$ eigenvalues

$$((z_1, z_1^{-1}), (z_2, z_2^{-1}), \ldots, (z_n, z_n^{-1})).$$

g usually conjugate to

$$(z_1, \ldots, z_n) \in GL(1, \mathbb{C})^n = H(\mathbb{C}) \subset \mathrm{Sp}(2n, \mathbb{C}).$$

$(z_1, \ldots, z_n)$ only determined up to permutation, inversions.

$H(\mathbb{C})$ is unique conjugacy class of Cartan in $\mathrm{Sp}(2n, \mathbb{C})$

Its Weyl group

$$W_{\mathbb{C}} = W(G(\mathbb{C}), H(\mathbb{C})) = N_{G(\mathbb{C})}(H(\mathbb{C}))/H(\mathbb{C}) = W(BC_n)$$

is called the nth hyperoctahedral group.

$W(BC_n) = S_n \rtimes (\pm 1)^n =$ permutations and inversions.

Real Cartan subgroups $\leftrightsquigarrow$ reality conditions on eigenvalues.

Each real Weyl group is a subgroup of $W(BC_n)$.

Cartans, eigenvalues, Weyl groups/$\mathbb{R}$

$g \in G(\mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})$ has $2n$ complex eigenvalues

$$((z_1, z_1^{-1}), (z_2, z_2^{-1}), \ldots, (z_n, z_n^{-1}))$$

permuted by complex conjugation.

Ways this happens $\leftrightsquigarrow$ expressions $n = a + 2b + c$:

1. $z_i = \overline{z}_i, \quad (1 \le i \le a)$;
2. $z_{a+2j-1} = \overline{z_{a+2j}}, \quad (1 \le j \le b)$; and
3. $z_{a+2b+k} = \overline{z_{a+2b+k}}^{-1}, \qquad (1 \le k \le c)$.

Conditions describe elts of $H_{a,b,c}(\mathbb{R}) = (\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b \times U(1)^c$.

$W_{a,b,c} = W(G(\mathbb{R}), H_{a,b,c}(\mathbb{R})) = W(BC_a) \times [W(BC_b) \rtimes (\pm 1)^b] \times S_c$.

Here $W(BC_b)$ acts simultaneously on (z_{a+2j-1}), $(\overline{z_{a+2j-1}})$.

$(\pm 1)^b$ interchanges some pairs $(z_{a+2j-1}, \overline{z_{a+2j-1}})$.

It's perhaps a surprise that the last factor is S_c (permutations) and not $W(BC_c)$ (which includes inversions).

Inverting some of the z_{a+2b+k} gives a group element conjugate by $G(\mathbb{C})$ but not by $G(\mathbb{R})$ (stably conjugate).

Distinction between conjugacy and stable conjugacy is source of multi-element L-packets in the Langlands classification.

Langlands classification

Theorem (Langlands)

$$\widehat{\mathrm{Sp}(2n,\mathbb{R})} = \coprod_{a+2b+c=n} \widehat{H_{a,b,c}(\mathbb{R})}_{\mathrm{reg}}/W_{a,b,c}.$$

$\widehat{H_{a,b,c}(\mathbb{R})} \leftrightsquigarrow \gamma \in \mathbb{C}^n$, $\epsilon \in (\mathbb{Z}/2\mathbb{Z})^a$, with

1. $\gamma_{a+2j-1} - \gamma_{a+2j} \in \mathbb{Z}, \quad 1 \le j \le b$, and
2. $\gamma_{a+2b+k} \in \mathbb{Z}, \quad 1 \le k \le c$.

Write γ as sum of continuous part (character of vector group A)

$$\nu = \Big(\gamma_1, \ldots, \gamma_a, \frac{\gamma_{a+1}+\gamma_{a+2}}{2}, \frac{\gamma_{a+1}+\gamma_{a+2}}{2}, \ldots, \frac{\gamma_{a+2b-1}+\gamma_{a+2b}}{2}, \frac{\gamma_{a+2b-1}+\gamma_{a+2b}}{2}, 0, \ldots, 0\Big) \in \mathbb{C}^{a+b} \subset \mathbb{C}^n$$

and discrete part (character of $T(\mathbb{R})_0$)

$$\lambda = \Big(0, \ldots, 0, \frac{\gamma_{a+1}-\gamma_{a+2}}{2}, \frac{-\gamma_{a+1}+\gamma_{a+2}}{2}, \ldots, \frac{\gamma_{a+2b-1}-\gamma_{a+2b}}{2}, \frac{-\gamma_{a+2b-1}+\gamma_{a+2b}}{2}, \gamma_{a+2b+1}, \ldots, \gamma_{a+2b+c}\Big) \in \mathbb{Z}^{b+c} \subset \left(\frac{1}{2}\mathbb{Z}\right)^n.$$

What about unitary representations?

Can define Herm dual V^h for $(\mathfrak{g}, K)$-module V.

Theorem (Knapp-Zuckerman). Suppose $G(\mathbb{R})$ real reductive, $H(\mathbb{R}) = T(\mathbb{R})A$ Cartan subgroup,

$$\gamma = (\lambda, \nu) \in \widehat{H(\mathbb{R})}_{\text{reg}}, \quad (\lambda \in \widehat{T(\mathbb{R})}, \nu \in \mathfrak{a}^*), \quad V = J(\gamma) \in \widehat{G(\mathbb{R})}.$$

1. $\gamma^h = (\lambda, -\overline{\nu})$; γ unitary $\iff \gamma = \gamma^h \iff \nu \in i\mathfrak{a}_0^*$.
2. $V^h \simeq J(\gamma^h)$; γ unitary $\iff J(\gamma)$ tempered.
3. V Herm $\iff V \simeq V^h \iff \gamma^h \in W(G(\mathbb{R}), H(\mathbb{R})) \cdot \gamma$.

Picture: $V \mapsto V^h$ is a complex conjugation on $\widehat{G(\mathbb{R})}$.

Hermitian reps = real points.

Easy real pts $\leftrightsquigarrow$ ν purely imaginary $\leftrightsquigarrow$ tempered reps.

Difficult real pts $\leftrightsquigarrow$ $-\overline{\nu} = w \cdot \nu \quad (w \in W(G(\mathbb{R}), H(\mathbb{R}))^\lambda)$.

Last cond is $\nu \in (i\mathfrak{a}_0^*)^w + (\mathfrak{a}_0^*)^{-w}$, real vec space of dimension $\dim A$.

Corollary (Knapp-Vogan). Each $V \in \widehat{G(\mathbb{R})}_h$ is unitarily induced from $V_L \otimes$ (unitary char) $\in \widehat{L(\mathbb{R})}_h$, with ν_L real.

What do the Langlands parameters mean?

Continuous part of Langlands param for $\mathrm{Sp}(2n, \mathbb{R})$ is

$$\nu_{a,b,c} = (z_1, \ldots, z_a, w_1/2, w_1/2, \ldots, w_b/2, w_b/2, 0, \ldots, 0),$$

with z_i and w_j complex; using the Weyl group we may assume z_i and w_j have nonnegative real part.

Rearrange these with decreasing real part as

$$\nu = (\nu_1, \cdots, \nu_n).$$

Then ν is a leading term in asymptotic expansions of matrix coefficients of $J(\lambda, \nu)$.

Discrete part of a Langlands param for $\mathrm{Sp}(2n, \mathbb{R})$ is

$$\lambda_{a,b,c} = (0, \ldots, 0, \ell_1/2, -\ell_1/2, \ldots, \ell_b/2, -\ell_b/2, n_1, \ldots, n_c),$$

with ℓ_j and n_k integers.

Rearrange these half integers in decreasing order as

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_n).$$

Then λ is close to the highest weight of the lowest representation of $U(n)$ appearing in $J(\lambda, \nu)$.

Looking closely at $K(\mathbb{R})$

Said for $\mathrm{Sp}(2n, \mathbb{R})$, disc part of Langlands parameter $\approx$ highest weight of lowest K-type.

To make this statement precise and more general, need to look closely at $\widehat{K(\mathbb{R})}$.

Reasons $\frac{\text{you don't know this already}}{\text{it's worth doing here}}$:

1. $K(\mathbb{R})$ is disconnected; Lie theorists are too lazy to talk about disconnected groups in grad courses.
2. Indexing $\widehat{K(\mathbb{R})}$ by highest weights is wrongheaded, persisting only for reasons cited in (1).
3. Construction of ρ_K covers that we'll use parallels details that I omitted from Langlands classification for reasons cited in (1).

Cartan subgroups of $K(\mathbb{R})$

Fix a maximal torus $T_{K,0}(\mathbb{R}) \subset K_0(\mathbb{R})$.

Fix pos roots $\Delta_K^+ \subset \Delta(\mathfrak{k}, T_{K,0}(\mathbb{R})) \leftrightsquigarrow$ Borel $\mathfrak{b}_K = \mathfrak{t}_K + \mathfrak{n}_K$.

Set $T_K(\mathbb{R}) = \mathrm{Norm}_{K(\mathbb{R})}(\mathfrak{b}_K)$, a large Cartan in $K(\mathbb{R})$.

OR fix Borel subgp $B_{K,0} \subset K_0$; define Borel subgp of K $B_K = N_K(B_{K,0})$.
Then $B_K \cap K(\mathbb{R}) = T_K(\mathbb{R}) =$ large Cartan in $K(\mathbb{R})$, $B_K = T_K N_K$.

$K(\mathbb{R})$ can be disconnected, exactly reflected in $T_K(\mathbb{R})$:

$$T_K(\mathbb{R})/T_{K,0}(\mathbb{R}) \simeq K(\mathbb{R})/K_0(\mathbb{R}).$$

Highest weight theory makes bijection

$$\widehat{K(\mathbb{R})} \longleftrightarrow \text{irreducible dominant reps of } T_K(\mathbb{R}).$$

For harmonic analysis, not the best parametrization.

Weyl dimension formula and Weyl character formula both use highest weight shifted by ρ_K.

Some easy covering groups

F a group: F-cover of group G is $1 \to F \to \widetilde{G} \to G \to 1$.

Easy exercise: F-cover is a contravariant functor.

Example. $F = \mu_n = n$th roots of 1, $1 \to \mu_n \to \mathbb{C}^\times \xrightarrow{n\text{th power}} \mathbb{C}^\times \to 1$.

Any character $\gamma \colon H \to \mathbb{C}^\times \rightsquigarrow n$th root of γ cover.

$$1 \to \mu_n \to \widetilde{H}_{\gamma/n} \to H \to 1, \quad \widetilde{H}_{\gamma/n} = \{(h,z) \in H \times \mathbb{C}^\times \mid \gamma(h) = z^n\}.$$

Representation τ of $\widetilde{H}_{\gamma/n}$ called genuine if $\tau(\omega) = \omega I \quad (\omega \in \mu_n)$.

$\widetilde{H}_{\gamma/n}$ has genuine character γ/n: $(\gamma/n)(h,z) = z$.

Proposition. $\otimes(\gamma/n)$ is a bijection $\widehat{H} \to \widehat{\left(\widetilde{H}_{\gamma/n}\right)}_{\text{genuine}}$.

General philosophical reason we need these: measures on manifold M $\leftrightsquigarrow$ line bundle $\bigwedge^{\dim M} T^*(M)$ (densities).

Hilbert spaces on $M \leftrightsquigarrow$ square roots of measures (half densities).

$M = G/H$: $\bigwedge^{\dim M} T^*(M) \leftrightsquigarrow$ char $\gamma \in \widehat{H} \quad (\gamma(h) = \det(\mathrm{Ad}(h)|_{\mathfrak{g}/\mathfrak{h}})^{-1})$.

half densities on $G/H \leftrightsquigarrow$ character $\gamma/2$.

ρ_K covers of Cartans in K

Recall Borel subgp $B_K = T_K N_K$ of K, T_K def over $\mathbb{R}$.

Get one diml character $2\rho_K \in \widehat{T_K}$, $2\rho_K(t) = \det(\mathrm{Ad}(t)|_{\mathfrak{b}_K}$.

$\rightsquigarrow$ (square root of $2\rho_K$) $= \rho_K$ cover $\widetilde{T_K}_{,\rho_K}$

Proposition. $\otimes\rho_K$ is bijection $\widehat{T_K} \to \left(\widetilde{T_K}_{,\rho_K}\right)^{\widehat{}}_{\text{genuine}}$; sends
(irr dom reps of T_K) $\leftrightsquigarrow$ (irr dom genuine regular reps of $\widetilde{T_K}_{,\rho_K}$).

Corollary. There is a bijection

$$\widehat{K} \longleftrightarrow \text{(irr dom regular reps of } \widetilde{T_K}_{,\rho_K}), \quad J_K(\gamma) \longleftrightarrow \gamma.$$

Suppose $\gamma_0 \in \mathfrak{t}^*$ is a weight of γ. Then

$$\dim(J_K(\gamma)) = \dim(\gamma) \cdot \prod_{\alpha \in \Delta_K^+} \frac{\langle \gamma_0, \alpha^\vee \rangle}{\langle \rho_K, \alpha^\vee \rangle}.$$

This is a formula for the Plancherel measure for $K(\mathbb{R})$.

Lowest K-types and Langlands parameters

Borel subgp $B_K = T_K N_K \subset K$, $\rho_K \in \mathfrak{t}_K^*$ half sum of roots.

Define $H_f(\mathbb{R})$ = cent in $G(\mathbb{R})$ of $T_{K,0}$, fundamental Cartan subgroup of $G(\mathbb{R})$.

Suppose γ irr dom genuine regular rep of $\widetilde{T_K}_{,\rho_K}$, so $J_K(\gamma) \in \widehat{K}$ has highest weight $\gamma - \rho_K$.

Fix $\gamma_1 \in i\mathfrak{t}_K(\mathbb{R})^*$ weight of γ.

Fix θ-stable pos $\Delta_G^+ \subset \Delta(\mathfrak{g}, \mathfrak{h}_f)$ so $\gamma_1 + \rho_K$ dom for Δ_G^+.

Define $2\rho_G^\vee$ = (sum of positive coroots for Δ_G^+) $\in i\mathfrak{t}_K(\mathbb{R})$.

Set height$(J_K(\gamma))$ = height$(\gamma) = \langle \gamma_1 + \rho_K, 2\rho^\vee \rangle$.

Lowest K-types of $V \in \widehat{G(\mathbb{R})}$ are $J_K(\gamma)$ of minimal height.

Theorem. Any lowest K-type $J_K(\gamma)$ of an irr rep $J(\lambda, \nu)$ determines the discrete Langlands parameter λ.

Assume $\gamma + \rho_K - \rho_G \in \left(\widetilde{T}_{f,\rho}\right)\widehat{}_{\text{genuine}}$ is dom reg for Δ_G^+. Then $H = H_f$, and $\lambda = \gamma + \rho_K - \rho_G$.

Recall that Δ_G^+ chosen to make $\gamma + \rho_K$ dominant. So hypothesis on $\gamma + \rho_K - \rho_G$ is always nearly true.

Discrete series lowest K-types

Fix $G(\mathbb{R})$ conn; assume $T_K(\mathbb{R}) \subset K(\mathbb{R})$ Cartan in $G(\mathbb{R})$.

Each $\Delta_G^+ \supset \Delta_K^+$ pos roots for T_K defines Weyl chamber

$$C_{\Delta_G^*} = \{\gamma \in i\mathfrak{t}_K(\mathbb{R})^* \mid \gamma(\alpha^\vee) \geq 0, \quad \alpha \in \Delta_G^+\},$$

closed convex cone in $i\mathfrak{t}_K(\mathbb{R})^*$.

Theorem (Hecht-Schmid). Suppose $\lambda \in \left(\widetilde{T}_{K,\rho}\right)\widehat{}_{\text{genuine}}$ is dom reg for Δ_G^+: HC param for a discrete series rep $J(\lambda)$.

1. Unique lowest K-type of $J(\lambda)$ is $J_K(\lambda + \rho_G - \rho_K)$.
2. Every K-type of $J(\lambda)$ is of the form $J_K(\lambda + \rho_G - \rho_K + S)$, $S =$ sum of roots in $\Delta_G^+ - \Delta_K^+$.

`https://1drv.ms/u/s!AuIZlbpNWacjgXYaZG6gUJimFt62`

has some pictures for $\mathrm{Sp}(4, \mathbb{R})$