K-theory of C^*-algebras

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Overview

Idea: study singular (e.g. non-Hausdorff) topological spaces through noncommutative C^*-algebras "associated to them".

To do this, generalise tools from algebraic topology to C^*-algebras.

This works well for K-theory.

I Topological K-theory

Topological K-theory of compact spaces

Let X be a compact Hausdorff space.

Definition

The **topological** K**-theory** of X is the abelian group $K^0(X)$ generated by the isomorphism classes of complex vector bundles on X, with the relation

$$[E \oplus F] \sim [E] + [F].$$

So

$$K^0(X) = \{[E] - [F]; E, F \to X \text{ complex vector bundles}\}.$$

The K-theory of a point

If $X = *$ is a point, then

$$K^0(*) = \{[\mathbb{C}^m] - [\mathbb{C}^n]; m, n \in \mathbb{Z}_{\geq 0}\} \cong \mathbb{Z}.$$

The Serre-Swan theorem

Theorem (Serre-Swan)

If $E \to X$ is a complex vector bundle over a compact Hausdorff space X, then there is a vector bundle $E' \to X$ such that for some n,

$$E \oplus E' \cong X \times \mathbb{C}^n.$$

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So $\Gamma(E)$ is a **finitely generated projective** $C(X)$**-module**:

$$\Gamma(E) \oplus \Gamma(E') = C(X)^n.$$

Conversely, every finitely generated projective $C(X)$-module is of this form. And for two vector bundles $E, F \to X$,

$$E \cong F \quad \Leftrightarrow \quad \Gamma(E) \cong \Gamma(F).$$

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So

$$K^0(X) = \{[M] - [N]; M, N \text{ finitely generated projective } C(X)\text{-modules}\}.$$

Projections

Let $E \to X$ be a complex vector bundle over a compact Hausdorff space X, and $E' \to X$ such that $E \oplus E' \cong X \times \mathbb{C}^n$. Let

$$p\colon X \to \mathrm{End}(X \times \mathbb{C}^n) = M_n(C(X))$$

be the orthogonal projection onto E, for a metric on $X \times \mathbb{C}^n$ such that $E \perp E'$.

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- $E = \mathrm{im}(p)$
- $\Gamma(E) = pC(X)^n = \{x \mapsto p(x)f(x); f \in C(X)^n\}$.

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- $E = \mathrm{im}(p)$
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Conversely, if $p\colon X \to M_n(C(X))$ is a projection, then $E = \mathrm{im}(p)$ is a vector bundle over X. And $\mathrm{im}(p) \cong \mathrm{im}(q)$ if and only if p and q are homotopic through projections, possibly in a larger matrix algebra. So

$$K^0(X) = \{[p] - [q]; p, q \in M_n(C(X)) \text{ for some } n \text{ are projections}\}.$$

Functoriality

Let X and Y be compact Hausdorff spaces, and $f\colon X \to Y$ continuous. Then f induces

$$f^*\colon K^0(Y) \to K^0(X)$$

by

- $f^*([E]-[F]) = [f^*E]-[f^*F]$ for vector bundles $E, F \to Y$
- $f^*([M]-[N]) = [M \otimes_{C(Y)} C(X)] - [N \otimes_{C(Y)} C(X)]$ for f.g.p. (right) $C(Y)$-modules M, N
- $f^*([p]-[q]) = [f^*p]-[f^*q]$ for projections $p, q \in M_n(C(Y))$, where

$$(f^*p)_{j,k}(x) = p_{j,k}(f(x)).$$

Locally compact spaces

Now let X be a locally compact Hausdorff space. Then its one-point compactification X^+ is a compact Hausdorff space.

Definition

The **topological** K-**theory** of X is the kernel $K^0(X)$ of the map

$$i^*\colon K^0(X^+) \to K^0(\infty) = \mathbb{Z}$$

induced by the inclusion $i\colon \infty \hookrightarrow X^+$ of the point at infinity.

Higher K-theory

Definition

Let X be a locally compact Hausdorff space. Then for $n \in \mathbb{Z}_{\geq 0}$,

$$K^n(X) = K^0(X \times \mathbb{R}^n).$$

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Theorem (Bott periodicity)

For all such X and n,

$$K^{n+2}(X) \cong K^0(X).$$

K-theory of C^*-algebras

K-theory of unital C^*-algebras

Let A be a C^*-algebra with a unit. Let

$$M_\infty(A) = \varinjlim_n M_n(A),$$

where $M_n(A) \hookrightarrow M_{n+1}(A)$ by padding with zeroes.

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Definition

The **even** K**-theory** of A is the abelian group $K_0(A)$ generated by homotopy classes of projections in $M_\infty(A)$, subject to the relation

$$[p \oplus q] \sim [p] \oplus [q].$$

Example

If X is a compact Hausdorff space,

$$K_0(C(X)) = K^0(X).$$

In particular, $K_0(\mathbb{C}) = \mathbb{Z}$.

Functoriality

Let A, B be a C^*-algebras with a units. Let $\varphi\colon A \to B$ be a $*$-homomorphism. Then

$$\varphi_*\colon K_0(A) \to K_0(B)$$

is defined by

$$\varphi([p] - [q]) = [\varphi_* p] - [\varphi_* q],$$

where

$$(\varphi_* p)_{j,k} = \varphi(p_{j,k}).$$

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Example

Let X and Y be compact Hausdorff spaces, and $f\colon X \to Y$ continuous. Then $f^*\colon C(Y) \to C(X)$ is a $*$-homomorphism, and $(f^*)_*$ is the map $f^*\colon K^0(Y) \to K^0(X)$ from earlier.

Note: contravariant vs. covariant functoriality.

Non-unital C^*-algebras

Let A be a C^*-algebra, not necessarily with a unit. The **unitisation** of A is the C^*-algebra

$$A^+ = A \oplus \mathbb{C},$$

with multiplication

$$(a+z)(b+w) = ab + wa + zb + zw,$$

involution

$$(a+z)^* = a^* + \bar{z}$$

and norm

$$\|a+z\| = \|a+z\|_{\mathcal{B}(A)}.$$

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Example

If X is a locally compact Hausdorff space, then $C_0(X)^+ = C(X^+)$.

K-theory of non-unital C^*-algebras

Let A be a C^*-algebra. Let $\pi\colon A^+ \to \mathbb{C}$ be projection onto the factor $\mathbb{C}$.

Definition

The **even K-theory** of A is the kernel of the map

$$\pi_*\colon K_0(A^+) \to K_0(\mathbb{C}) = \mathbb{Z}.$$

Higher K-groups

Definition

For a C^*-algebra A and $n \in \mathbb{Z}_{\geq 0}$,

$$K_n(A) = K_0(A \otimes C_0(\mathbb{R}^n)).$$

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Theorem (Bott periodicity)

For all such A and n,

$$K_{n+2}(A) = K_n(A).$$

We sometimes write $K_*(A) := K_0(A) \oplus K_1(A)$.

Stability and continuity

Theorem (Stability)

If two C^*-algebras A and B are Morita equivalent, then for all n,

$$K_n(A) = K_n(B).$$

In particular, $K_n(A \otimes \mathcal{K}) = K_n(A)$.

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Theorem (Continuity)

If $(A_j)_{j=1}^{\infty}$ is a sequence of C^*-algebras connected by $*$-homomorphisms $A_n \to A_{n+1}$, then

$$\varinjlim_j K_n(A_j) = K_n(\varinjlim_j A_j).$$

Direct sums

For a sequence of C^*-algebras $(A_j)_{j=1}^{\infty}$, let

$$\bigoplus_{j=1}^{\infty} A_j$$

be the completion of the algebraic direct sum in the norm

$$\|(a_1, a_2, \ldots, a_n, 0, \ldots)\| = \sup_j \|a_j\|_{A_j}.$$

Lemma

We have

$$K_n\Big(\bigoplus_{j=1}^{\infty} A_j\Big) = \bigoplus_{j=1}^{\infty} K_n(A_j).$$

Proof.

This is elementary for finite direct sums. By continuity, this extends to infinite direct sums. □

Homotopy invariance

Let A and B be two C^*-algebras.

Definition

Two $*$-homomorphisms $f, g\colon A \to B$ are **homotopic** if there is a path of $*$-homomorphisms $(f_t)_{t\in[0,1]}\colon A \to B$ such that $f_0 = f$, $f_1 = g$ and for all $a \in A$,

$$t \mapsto f_t(a)$$

is a continuous path in B.

Proposition

If $f, g\colon A \to B$ are homotopic, then they induce the same map

$$f_* = g_*\colon K_*(A) \to K_*(B).$$

The six-term exact sequence

Theorem

Let A be a C^*-algebra, and $J \subset A$ a closed, $*$-closed, two-sided ideal. Then there is an exact sequence

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \delta \uparrow & & & & \downarrow \delta \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J) \end{array}$$

Here we use Bott periodicity: $K_2(J) = K_0(J)$.

Examples

- For any locally compact Hausdorff space X,

$$K_n(C_0(X)) = K^n(X).$$

- $$K_n(C_0(\mathbb{R}^n)) = \mathbb{Z} \qquad K_{n+1}(C_0(\mathbb{R}^n)) = 0$$

- $$K_0(C_0([0,\infty))) = K_1(C_0([0,\infty))) = 0$$

- $$K_0(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2)) = \mathbb{Z} \qquad K_1(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2)) = 0$$

More examples

- For any Hilbert space H,

$$K_0(\mathcal{K}(H)) = K_0(\mathbb{C}) = \mathbb{Z} \qquad K_1(\mathcal{K}(H)) = K_1(\mathbb{C}) = 0.$$

- For any infinite-dimensional, separable Hilbert space H,

$$K_0(\mathcal{B}(H)) = K_1(\mathcal{B}(H)) = 0.$$

- If K is a compact Lie group, then by Peter–Weyl,

$$K_0(C^*(K)) = K_0\Big(\bigoplus_{V \in \hat{K}} \mathrm{End}(V)\Big) = \bigoplus_{V \in \hat{K}} K_0\big(\mathrm{End}(V)\big) = \bigoplus_{V \in \hat{K}} \mathbb{Z} = R(K).$$

K_1 via invertible matrices

If A is a unital C^*-algebra, let $\mathrm{GL}_n(A)$ be the group of invertible $n \times n$ matrices over A. We embed $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A)$ by adding a 1 on the bottom-right corner, and zeros everywhere else. Set

$$\mathrm{GL}_\infty(A) = \varinjlim_n \mathrm{GL}_n(A).$$

Let $\mathrm{GL}_\infty(A)_0 < \mathrm{GL}_\infty(A)$ be the connected component of the identity.

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Proposition

For every, possibly non-unital, C^*-algebra A, there is an isomorphism of abelian groups

$$\mathrm{GL}_\infty(A^+)/\,\mathrm{GL}_\infty(A^+)_0 \cong K_1(A).$$

III K-theory of C^*-algebras of reductive Lie groups

The case of a real reductive group

For a real reductive Lie group G, the group C^*-algebra $C^*_r(G)$ can be described explicitly via representation theory. (See Tyrone's lectures.)

This can be used to describe $K_0(C^*_r(G))$, but it gives more information.

The reduced C^*-algebra of a real reductive group

Let G be a connected, linear, real reductive Lie group.

Theorem (A. Wassermann, Clare–Crisp–Higson, Clare–Higson–Song–Tang)

The reduced group C^*-algebra of G is Morita equivalent to

$$\sum_{[P,\sigma]} C_0(\mathfrak{a}/W'_\sigma) \rtimes R_\sigma.$$

- $P = MAN$ runs over the cuspidal parabolics
- σ is a discrete series representation of M
- $[P,\sigma] = [P',\sigma']$ if there is a $k \in K$ such that $M'A' = kMAk^{-1}$ and $\mathrm{Ad}^*_k \sigma' \cong \sigma$
- $W_\sigma = \{w \in N_K(MA); \mathrm{Ad}^*_w \sigma \cong \sigma\}/(K \cap M)$
- $I(w,\sigma)\colon \mathrm{Ind}_P^G(\sigma \otimes 1) \to \mathrm{Ind}_P^G(\mathrm{Ad}^*_w \sigma \otimes 1)$ is the Knapp–Stein intertwiner
- $W'_\sigma = \{w \in W_\sigma; I(w,\sigma) \in \mathbb{C}I\}$
- $W_\sigma = W'_\sigma \rtimes R_\sigma$ for the R-group $R_\sigma \cong (\mathbb{Z}/2)^n$.

The K-theory $C_r^*(G)$: trivial contributions

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The K-theory $C^*_r(G)$: trivial contributions

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Lemma

If $W'_\sigma \neq \{e\}$, then

$$K_*(C_0(\mathfrak{a}/W'_\sigma) \rtimes R_\sigma) = 0.$$

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Example

If $\dim(\mathfrak{a}) = 1$ and $W_\sigma = W'_\sigma$ acts by reflections,

$$K_*(C_0(\mathfrak{a}/W'_\sigma) \rtimes R_\sigma) = K_*(C_0([0,\infty))) = 0.$$

The K-theory $C_r^*(G)$: nontrivial contributions

Theorem (Knapp–Stein)

If $W'_\sigma = \{e\}$, then $R_\sigma = (\mathbb{Z}/2)^{\dim(A_{\max}) - \dim(A)}$, and $\mathfrak{a}$ is R_σ-equivariantly isomorphic to $\mathbb{R}^{\dim(A)}$, on which R_σ acts by reflections in the first $\dim(A_{\max}) - \dim(A)$ coordinates.

The K-theory $C_r^*(G)$: nontrivial contributions

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Corollary

If $W'_\sigma = \{e\}$, then

$$C_0(\mathfrak{a}/W'_\sigma) \rtimes R_\sigma \cong C_0(\mathbb{R}^{\dim(A_{\max})}) \otimes \left(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2)\right)^{\otimes(\dim(A_{\max})-\dim(A))},$$

and

$$K_{\dim(A_{\max})}\left(C_0(\mathfrak{a}/W'_\sigma) \rtimes R_\sigma\right) = \mathbb{Z}$$
$$K_{\dim(A_{\max})+1}\left(C_0(\mathfrak{a}/W'_\sigma) \rtimes R_\sigma\right) = 0.$$

Note: $\dim(A_{\max}) = \dim(G/K) \mod 2$.

The K-theory of $C^*_r(G)$

Conclusion:

$$K_{\dim(G/K)}(C^*_r(G)) = \bigoplus_{[P,\sigma]} K_{\dim(G/K)}(C_0(\mathfrak{a}/W'_\sigma) \rtimes R_\sigma)$$
$$= \bigoplus_{[P,\sigma], W'_\sigma = \{e\}} \mathbb{Z}$$

and

$$K_{\dim(G/K)+1}(C^*_r(G)) = 0.$$

The discrete series

Suppose that G has a compact Cartan subgroup. Then it is a cuspidal parabolic of itself, and $C_r^*(G)$ has the direct summand

$$\bigoplus_{\sigma \in \hat{G}_{\mathrm{ds}}} \mathcal{K}(H_\sigma).$$

So $K_0(C_r^*(G))$ has the direct summand

$$\bigoplus_{\sigma \in \hat{G}_{\mathrm{ds}}} K_0(\mathcal{K}(H_\sigma)) = \bigoplus_{\sigma \in \hat{G}_{\mathrm{ds}}} \mathbb{Z}.$$

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The generator corresponding to $\sigma \in \hat{G}_{\mathrm{ds}}$ can be described as follows. Fix $v \in H_\sigma$ of norm 1. Consider the matrix coefficient

$$m_{v,v}(g) = (\sigma(g)v, v).$$

Then $p_\sigma := d_\sigma m_{v,v}$ is an idempotent in $C_r^*(G)$, and its class generates $K_0(\mathcal{K}(H_\sigma))$.

Example: $G = \mathrm{SL}(2, \mathbb{R})$

If $G = \mathrm{SL}(2, \mathbb{R})$, then we have a Morita equivalence

$$C^*_r(G) \sim \underbrace{\overbrace{C_0(\mathbb{R}/(\mathbb{Z}/2))}^{\text{spherical princ. ser.}} \oplus \overbrace{C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2)}^{\text{non-spherical princ. ser.}}}_{\text{minimal parabolic}} \oplus \underbrace{\bigoplus_{n=1}^{\infty} \overbrace{\mathcal{K}(H_{n,+})}^{\text{hol. d.s.}} \oplus \overbrace{\mathcal{K}(H_{n,-})}^{\text{antihol. d.s.}}}_{\text{maximal parabolic } G}$$

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So

$$K_0(C^*_r(G)) = K_0\big(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2)\big) \oplus \bigoplus_{n=1}^{\infty} K_0(\mathcal{K}(H_{n,+})) \oplus K_0(\mathcal{K}(H_{n,-})),$$

where every K-group on the right hand side is $\mathbb{Z}$.

Example: complex groups

Let G be a complex semisimple Lie group. Then there is one parabolic $P = MAN$ up to conjugacy, M is a torus, and all principal series representations are irreducible. So $W'_\sigma = W_\sigma$ for all $\sigma \in \hat{M}$, and there is a Morita equivalence

$$C_r^*(G) \sim \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}/W_\sigma) = C_0(\hat{G}_{\text{temp}}).$$

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$$C^*_r(G) \sim \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}/W_\sigma) = C_0(\hat{G}_{\text{temp}}).$$

So

$$K_n(C^*_r(G)) = K^n(\hat{G}_{\text{temp}}) = \bigoplus_{W_\sigma = \{e\}} K^n(\mathfrak{a}),$$

and

$$K^{\dim(G/K)}(\mathfrak{a}) = \mathbb{Z} \qquad K^{\dim(G/K)+1}(\mathfrak{a}) = 0.$$

This case was worked out by Penington and Plymen in 1983.

IV The higher index

The Fredholm index

Consider the ideal $\mathcal{K}(H) \subset \mathcal{B}(H)$. We have the boundary map in the six-term exact sequence

$$\delta\colon K_1(\mathcal{B}(H)/\mathcal{K}(H)) \to K_0(\mathcal{K}(H)) = \mathbb{Z}.$$

The Fredholm index

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$$\delta\colon K_1(\mathcal{B}(H)/\mathcal{K}(H)) \to K_0(\mathcal{K}(H)) = \mathbb{Z}.$$

Let $F \in \mathcal{B}(H)$ be Fredholm. Then its class in $\mathcal{B}(H)/\mathcal{K}(H)$ is invertible, and defines an element

$$[F] \in K_1(\mathcal{B}(H)/\mathcal{K}(H)).$$

Proposition

Now

$$\delta[F] = \mathrm{index}(F) = \dim(\ker(F)) - \dim(H/\operatorname{im}(F)).$$

The abstract index

Whenever we have an ideal $J \subset A$ and an element $a \in A$ invertible modulo J, we can **define**

$$\mathrm{index}_J(a) := \delta[a] \in K_0(J),$$

where

$$\delta\colon K_1(A/J) \to K_0(J)$$

is the boundary map form the six-term exact sequence.

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where

$$\delta \colon K_1(A/J) \to K_0(J)$$

is the boundary map form the six-term exact sequence.

In particular, if $\mathcal{E}$ is a Hilbert A-module, then an A-Fredholm operator $F \in \mathcal{L}(\mathcal{E})$ has an index

$$\mathrm{index}_{\mathcal{K}(\mathcal{E})}(F) \in K_0(\mathcal{K}(\mathcal{E})).$$

If $\mathcal{E}$ is **full**, i.e.

$$\mathrm{span}\{(v, w); v, w \in \mathcal{E}\} \subset A$$

is dense, then $\mathcal{E}$ defines a Morita equivalence between $\mathcal{K}(\mathcal{E})$ and A, and we obtain

$$\mathrm{index}_{\mathcal{K}(\mathcal{E})}(F) \in K_0(A).$$

The equivariant index

Let G be a locally compact group acting properly and isometrically on a Riemannian manifold M, such that M/G is compact. Let D be a G-equivariant elliptic, self-adjoint, odd-graded, first order differential operator on a $\mathbb{Z}/2$-graded Hermitian G-vector bundle $E \to M$.

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We constructed a Hilbert $C^*(G)$-module $\mathcal{E}$ by completing $\Gamma_c(E)$ in the inner product

$$(s_1, s_2)(g) = (s_1, g \cdot s_2)_{L^2(E)}.$$

The operator D defines a $C^*(G)$-Fredholm operator $F \in \mathcal{L}(\mathcal{E})$, which is a modification of

$$\frac{D}{\sqrt{D^2+1}}.$$

Here $C^*(G)$ can be either the maximal or reduced group C^*-algebra.

Definition

The **equivariant index** of D is

$$\mathrm{index}_G(D) = \mathrm{index}_{\mathcal{K}(\mathcal{E})}(F) \in K_0(C^*(G)).$$

Relation with classical indices

If $G = K$ is **compact**, then

$$\mathrm{index}_K(D) = [\ker(D)] - [\ker(D^*)] \in K_0(C^*(K)) = R(K).$$

(This is the Fredholm index if $K = \{e\}$.)

Relation with classical indices

If $G = K$ is **compact**, then

$$\mathrm{index}_K(D) = [\ker(D)] - [\ker(D^*)] \in K_0(C^*(K)) = R(K).$$

(This is the Fredholm index if $K = \{e\}$.)

Suppose that M is the universal cover of a compact manifold X, and that $G = \Gamma = \pi_1(X)$. Let D_X be the operator on $E/\Gamma \to X$ induced by D. Consider the homomorphism

$$\sum_\Gamma : C^*_{\mathrm{max}}(\Gamma) \to \mathbb{C}$$

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$$\sum_\Gamma \colon C^*_{\mathrm{max}}(\Gamma) \to \mathbb{C}$$

given by summing functions over Γ. Then

$$(\sum_\Gamma)_* \, \mathrm{index}_\Gamma(D) = \mathrm{index}(D_X) \quad \in \mathbb{Z}.$$

So $\mathrm{index}_\Gamma(D)$ is a **refinement** of $\mathrm{index}(D_X)$. (The same can be achieved via the trace $\tau(f) = f(e)$ on $C^*(\Gamma)$ via Atiyah's L^2-index theorem.)

Dirac operators on G/K

Let G be a connected Lie group. Let $K < G$ be maximal compact, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ an orthogonal decomposition for a K-invariant inner product. Suppose (for simplicity) that the map

$$\mathrm{Ad}\colon K \to \mathrm{SO}(\mathfrak{p})$$

lifts to

$$\widetilde{\mathrm{Ad}}\colon K \to \mathrm{Spin}(\mathfrak{p}).$$

Then we view the standard representation $S_{\mathfrak{p}}$ of $\mathrm{Spin}(\mathfrak{p})$ as a representation of K.

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Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis of $\mathfrak{p}$. For $V \in \hat{K}$, consider the Dirac operator

$$D^V_{G/K} = \sum_{j=1}^{n} L_{X_j} \otimes c(X_j) \otimes 1_V$$

on

$$\left(C^\infty(G) \otimes S_{\mathfrak{p}} \otimes V\right)^K.$$

The discrete series

Theorem (Parthasarathy 1972, Atiyah–Schmid 1977)

Let G be a connected, real semisimple Lie group with $\mathrm{rank}(G) = \mathrm{rank}(K)$*. Let $V \in \hat{K}$, and let λ be its highest weight. Then*

- *if $\lambda + \rho_K$ is regular, then $\ker_{L^2}(D^V_{G/K})$ is the discrete series representation of G with Harish–Chandra parameter $\lambda + \rho_K$*
- *if $\lambda + \rho_K$ is singular, then $\ker_{L^2}(D^V_{G/K}) = 0$.*

Theorem (Connes–Moscovici 1982)

Let G be a connected, real semisimple Lie group with $\mathrm{rank}(G) > \mathrm{rank}(K)$*. Then for all $V \in \hat{K}$, $\ker_{L^2}(D^V_{G/K}) = 0$.*

The Connes–Kasparov conjecture

Using the equivariant index, we obtain information from $D^V_{G/K}$ for **any** G and V.

Definition

Dirac induction is the map

$$\text{D-Ind}_K^G \colon R(K) \to K_*(C_r^*(G))$$

given by $\text{D-Ind}_K^G[V] = \text{index}_G(D^V_{G/K})$.

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Conjecture (Connes–Kasparov)

For almost connected Lie groups G, Dirac induction is an isomorphism of abelian groups.

Proofs of the Connes–Kasparov conjecture

Conjecture (Connes–Kasparov)

For almost connected Lie groups G, Dirac induction $V \mapsto \mathrm{index}_G(D^V_{G/K})$ is an isomorphism of abelian groups $R(K) \cong K_*(C^*_r(G))$.

Proofs:

- for semisimple/reductive groups:
  - by computing $C^*_r(G)$: A. Wassermann 1987 + recent work by Clare, Crisp, Higson, Song, Tang and Vogan
  - directly: V. Lafforgue 2002, Afgoustidis 2019
- in general: Chabert–Echterhoff–Nest 2003.

Example: the discrete series

Suppose that G has discrete series representations, and let π be a discrete series representation. We saw that π contributes a generator

$$[p_\sigma] := [d_\sigma m_{v,v}] \in K_0(\mathcal{K}(H_\sigma)) \subset K_0(C^*_r(G)).$$

This generator equals

$$[p_\sigma] = \text{D-Ind}_K^G[V] = \text{index}_G(D^V_{G/K}),$$

with $D^V_{G/K}$ as in Parthasarathy's/Atiyah–Schmid's construction.

Classifying space for proper actions

Let G be a locally compact, Hausdorff, second countable group.

Definition

A **classifying space for proper actions** by G is a topological space $\underline{E}G$ with a proper action by G, such that for any proper G-space X,

- there is a G-equivariant continuous map $X \to \underline{E}G$
- any two such maps are G-equivariantly homotopic.

This exists and is unique up to G-equivariant homotopy equivalence.

Example

If G is a connected Lie group, we can take $\underline{E}G = G/K$ by Abels' theorem.

The Baum–Connes conjecture

Conjecture (Baum–Connes 1982)

The equivalent index defines an isomorphism of abelian groups

$$RK_*^G(\underline{E}G) \to K_*(C_r^*(G)),$$

where RK_*^G *denotes* **representable equivariant K-homology**.

Intuitively, $RK_*^G(\underline{E}G)$ consists of homotopy classes of abstract G-equivariant elliptic operators on proper G-spaces X such that X/G is compact.

The idea is that this is more computable than $K_*(C_r^*(G))$.

The Baum–Connes conjecture

Conjecture (Baum–Connes 1982)

The equivariant index defines an isomorphism of abelian groups

$$RK_*^G(\underline{E}G) \to K_*(C_r^*(G)),$$

where RK_*^G *denotes* **representable equivariant K-homology**.

Intuitively, $RK_*^G(\underline{E}G)$ consists of homotopy classes of abstract G-equivariant elliptic operators on proper G-spaces X such that X/G is compact.

The idea is that this is more computable than $K_*(C_r^*(G))$.

The conjecture has been proved for many classes of groups.

An important open case is non-cocompact lattices in Lie groups with real rank ≥ 2, e.g. $\mathrm{SL}(3, \mathbb{Z})$.

Special cases

- If G is a connected, real Lie group, then

$$RK_G^*(\underline{E}G) = K_G^*(G/K) = R(K)$$

  via $[D_{G/K}^V] \leftrightarrow [V]$, and the Baum–Connes conjecture becomes the Connes–Kasparov conjecture.
- The Novikov conjecture in manifold topology (rational injectivity), where $G = \pi_1(X)$ as earlier.

Topics not covered—a non-exhaustive list

- KK-theory: $KK(A, B)$, encodes relations/maps between $K_*(A)$ and $K_*(B)$
- extracting information from K-theory via pairing with cyclic cohomology or traces, e.g. orbital integrals

Thank you