C^*-algebras of real reductive groups

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AIM RTNCG: C^*-Algebras for Representation Theorists

7 & 14 June 2021

Our goal

G : real reductive group (eg $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, etc)

$C_r^*(G)$: norm closure of $C_c(G)$ in $\mathrm{B}(L^2 G)$

Goal: Compute $C_r^*(G)$ in some useful way.

Theorem [A. Wassermann]: $C_r^*(G) \underset{\text{Morita}}{\sim} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma$

— What does 'useful' mean? Eg: a way that helps us compute K-theory (see Peter Hochs's second lecture)

— Why $C_r^*(G)$ and not $C^*(G)$? $\widehat{C^*(G)} \cong \widehat{G}_{\mathrm{u}}$, $\widehat{C_r^*(G)} \cong \widehat{G}_{\mathrm{temp}}$. So $C_r^*(G)$ is much easier to understand than $C^*(G)$.

— How is $C_r^*(G)$ different to $\mathcal{C}(G)$, $\mathcal{S}(G)$, ...? C_0 functions are much easier to work with than, eg, Schwartz functions or C_c^∞ functions.

Some resources

For this C^*-algebra computation:

- — A. Wassermann [Comptes Rendus, 1987]
- — Penington-Plymen [JFA 1983]
- — Plymen [JFA, 1990]
- — Leung-Plymen [Compositio, 1991]
- — Clare-Crisp-Higson [Compositio, 2016]
- — Afgoustidis-Aubert [arXiv:2002.12864]

Representation theory: We'll use lots of deep RT theorems [Harish-Chandra, Langlands, Knapp-Stein,...]. Read about them here:

- — Knapp [Overview, 1986]
- — Wallach [RRGs vols 1&2, 1988&1992]

C^*-algebras: We'll use mostly 'classical' theory [Gelfand-Naimark, Segal, Kadison, Kaplansky, ...]. Almost everything we'll need is in

- — Dixmier [C^*-algèbres, 1964; C^*-algebras, 1977]

C^*-algebra reminders

— A C^*-algebra is a $\mathbb{C}$-algebra with a complete norm and an involution $*$ (conjugate-linear, $(ab)^* = b^*a^*$), with $\|a^*a\| = \|a\|^2$.

— A homomorphism of C^*-algebras is a linear map compatible with multiplication and involution.

Theorem: Every homomorphism of C^*-algebras is continuous and has closed range. Every isomorphism of C^*-algebras is isometric.

Examples: • $C_0(X)$: continuous functions vanishing at ∞

• $\mathrm{B}(H)/\mathrm{K}(H)$: bounded/compact operators on a Hilbert space H

• $C_0(X, \mathrm{K}(H))$: operator-valued C_0 functions

• Direct sums: $\bigoplus_i A_i = \{(a_i) \mid \|a_i\| \to 0 \text{ as } i \to \infty\}$

• $C^*(G)$ and its quotient $C^*_r(G)$, for G a locally compact group

— C^*-algebras often don't have 1 (eg $C_0(X)$, $\mathrm{K}(H)$) ... and that's often not a problem (approximate identities; adjoining an identity; multiplier algebras).

C^*-algebra reminders, continued

— Representation of A : homomorphism $\pi : A \to \mathrm{B}(H)$

Examples: • $\mathrm{K}(H)$ has only one irrep, namely H

• G real reductive: irreps of $C_r^*(G) \longleftrightarrow \widehat{G}_{\mathrm{temp}}$ [Harish-Chandra]

— A is liminal (CCR) if $\pi(A) = \mathrm{K}(H)$ for every irrep π.

Examples of liminal C^*-algebras: $C_0(X)$, $\mathrm{K}(H)$, $C_0(X, \mathrm{K}(H))$, ...

Theorem [Harish-Chandra]: G real reductive $\Longrightarrow$ $C_r^*(G)$ liminal.

— GNS construction: reps of C^*-algebras come from states (positive linear functionals of norm 1).

Then Hahn-Banach and Krein-Milman $\Longrightarrow$

• the irreps of a C^*-algebra separate points

• irreps of a subalgebra $B \subseteq A$ extend to irreps of A

• Subalgebras of liminal algebras are liminal.

A Stone-Weierstrass theorem

Theorem [Kaplansky]: Suppose A liminal, $B \subseteq A$ a C^*-subalgebra, such that

— each irrep of A remains irreducible on B, and

— inequivalent irreps of A remain inequivalent on B.

Then $B = A$.

(This holds more generally: eg if A, B are separable and nuclear [Sakai].)

Examples: • Commutative case: Stone-Weierstrass

• $\mathrm{K}(H)$ has no proper irreducible subalgebra (this is an ingredient in the proof, rather than a corollary)

• $A = C_0(X, \mathrm{K}(H))$: If a subalgebra $B \subseteq A$ is a $C_0(X)$-submodule, and if the point evals $b \mapsto b(x)$ are all surjective, then $B = A$.

Kaplansky on CCR (liminal) C^*-algebras

6. CCR-algebras

I shall briefly describe how CCR-algebras started, and where they stand today. The material of §5 more or less covers the case where, for every primitive ideal P in the C^*-algebra A, A/P is finite-dimensional. At just about the time this work was completed, the important papers of Gelfand and Naimark on representations of semi-simple Lie groups were beginning to appear. One aspect of their results was the following: for the relevant C^*-algebras every A/P was the algebra of completely continuous operators on a Hilbert space. This encouraged me to see whether some progress was possible on this class of C^*-algebras. The unimaginative name CCR stands for "completely continuous representations". I stand ready to yield to Dixmier's "liminaire", except that I confess that I do not know what it means. At any rate, it turned out that the fibre bundle type of result survived, when appropriately formulated.

[CBMS lecture notes, 1970]

Real reductive groups

A real reductive group is (for us) a closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{R})$, closed under transpose, that is the group of real points of a connected reductive algebraic group defined over $\mathbb{R}$.

Examples: $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{C})$, ...

Write $G = M_G \times A_G$ where A_G is the split component and $M_G = {}^0G = \{g \in G \mid \chi(g) = \pm 1 \text{ for all homs } \chi : G \to \mathbb{R}^\times\}$.

Examples: • $G = \mathrm{SL}_n(\mathbb{R})$, $M_G = G$, $A_G = \{1\}$

• $G = \mathrm{GL}_n(\mathbb{R})$, $M_G = \{\det g = \pm 1\}$, $A_G = \{a \cdot 1 \mid a > 0\}$

Note that $\exp : \mathfrak{a}_G \xrightarrow{\cong} A_G$ is a group isomorphism.

Discrete series

$$G = M_G A_G \qquad \mathfrak{a}_G \cong A_G \implies \mathfrak{a}_G^* \cong \widehat{A_G}: \quad \chi \longmapsto \exp(i\chi(\ \ ))$$

Let $\sigma : M_G \to \mathrm{U}(H_\sigma)$ be irreducible and square-integrable.

$\rightsquigarrow \sigma \otimes \chi : G \to \mathrm{U}(H_\sigma)$, $ma \mapsto \sigma(m)\chi(a)$, for each $\chi \in \mathfrak{a}_G^*$

$\rightsquigarrow \sigma \otimes \chi : C_r^*(G) \to \mathrm{B}(H_\sigma)$, $f \mapsto \int_G f(g)(\sigma \otimes \chi)(g)\, dg$

Lemma: $\pi_{G,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_G^*, \mathrm{K}(H_\sigma))$, $\pi_{G,\sigma}(f)(\chi) = \sigma \otimes \chi(f)$ is a surjective homomorphism of C^*-algebras.

Proof: • $\sigma \otimes \chi(f) \in \mathrm{K}(H_\sigma)$: Schur orthogonality on M_G

• $\pi_{G,\sigma}(f) \in C_0$: Fourier transform on A_G

• surjectivity: use Stone-Weierstrass. The $\sigma \otimes \chi$ are irreducible (they are on M_G) and pairwise inequivalent (they are on A_G). □

Discrete series, continued

$G = M_G A_G$, $\sigma \in \widehat{M}_{L^2} \rightsquigarrow \pi_{G,\sigma} : C_r^*(G) \twoheadrightarrow C_0(\mathfrak{a}_G^*, \mathrm{K}(H_\sigma))$

Now consider all square-integrable σ at the same time:

Theorem: $\bigoplus \pi_{G,\sigma} : C_r^*(G) \to \bigoplus_{\sigma \in \widehat{M}_{L^2}} C_0(\mathfrak{a}_G^*, \mathrm{K}(H_\sigma))$ is a surjective $*$-homomorphism

Proof: Surjectivity follows as before from Stone-Weierstrass.

Question: why does $C_r^*(G)$ map into the direct sum? (Recall that $\bigoplus_i A_i := \{(a_i) \mid \|a_i\| \to 0\}$)

Enter our first major input from representation theory:

Theorem [Harish-Chandra]: Each irreducible representation of K (maximal compact in G) appears in only finitely many σ.

The C^*-algebra theorem follows by considering the dense subspace of K-finite elements of $C_r^*(G)$. □

Parabolic subgroups

G : real reductive group

$P = L_P N_P = M_P A_P N_P$: a parabolic subgroup of G

Example: In $G = \mathrm{GL}_3(\mathbb{R})$:

$$P = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right] \quad L_P = \left[\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline 0 & 0 & * \end{array}\right] \quad N_P = \left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$A_P = \left[\begin{array}{cc|c} a & 0 & 0 \\ 0 & a & 0 \\ \hline 0 & 0 & b \end{array}\right] \qquad M_P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{m}} & 0 \\ & & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right]$$

$$(a, b > 0) \qquad (\det m = \pm 1)$$

Parabolic subgroups, continued

Parabolic subgroup $P = L_P N_P = M_P A_P N_P \subseteq G$

Properties:

— L_P is a real reductive group

— $P \cong L_P \rtimes N_P$

— $L_P \cong M_P \times A_P$

— $\exp : \mathfrak{a}_P \xrightarrow{\cong} A_P$ ($\Longrightarrow \mathfrak{a}_P^* \cong \widehat{A_P}$)

— $G = KP$ for maximal compact K (note that $K \cap P \neq \{1\}$)

— There are only finitely many P, up to conjugacy

Parabolic induction

$P = M_P A_P N_P$ parabolic subgroup of G

$\sigma \in \widehat{M}_{L^2}$, $\chi \in \mathfrak{a}_P^* \rightsquigarrow \sigma \otimes \chi \in \widehat{P}$: $\sigma \otimes \chi(man) = \sigma(m)\chi(a)$

Parabolic induction: $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ is the unitary representation of G induced from $\sigma \otimes \chi$.

We use the compact picture of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$:

$G = KP \implies \mathrm{Ind}_P^G(\sigma \otimes \chi) \cong \mathrm{Ind}_{K \cap M_P}^K(\sigma)$ over K

$\implies$ all $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ can be realised on the same space $\mathrm{Ind}_P^G H_\sigma$

Lemma: $\pi_{P,\sigma}(f)(\chi) := \mathrm{Ind}_P^G(\sigma \otimes \chi)(f)$ defines a C^*-algebra homomorphism $\pi_{P,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$. □

Goal: characterise the image of $\pi_{P,\sigma}$.

Intertwining operators

Fix P and $\sigma \in \widehat{M}_{L^2} \rightsquigarrow \pi_{P,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$

When $P = G$: the $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ are irreducible and pairwise inequivalent $\Longrightarrow \pi_{G,\sigma}$ is surjective.

When $P \neq G$ there are nontrivial intertwining operators between the $\mathrm{Ind}_P^G(\sigma \otimes \chi)$, and $\pi_{P,\sigma}$ is not surjective.

'Weyl' groups: • $W_P := \mathrm{Norm}_G(A_P)/\mathrm{Cent}_G(A_P)$; a finite group, acting by conjugation on $\mathfrak{a}_P^*$ and on $\widehat{M_P}$.

- For each $\sigma \in \widehat{M}_{L^2}$: $W_\sigma := \{w \in W_P \mid w\sigma \cong \sigma\}$
- For each $\chi \in \mathfrak{a}_P^*$: $W_{\sigma,\chi} := \{w \in W_\sigma \mid w\chi = \chi\}$

Intertwining operators, continued

Fix P and $\sigma \in \widehat{M}_{L^2} \rightsquigarrow \pi_{P,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$

$W_\sigma = \{w \in W_P \mid w\sigma \cong \sigma\} \quad W_{\sigma,\chi} = \{w \in W_\sigma \mid w\chi = \chi\}$

Theorem [Knapp-Stein, Harish-Chandra]: There are unitary operators $I(w,\chi) \in \mathrm{U}(\mathrm{Ind}_P^G H_\sigma)$ $(w \in W_\sigma,\ \chi \in \mathfrak{a}_P^*)$ satisfying:

- $\chi \mapsto I(w,\chi)$ is strongly continuous $\mathfrak{a}_P^* \to \mathrm{U}(\mathrm{Ind}_P^G H_\sigma)$
- $\mathrm{Ind}_P^G(\sigma \otimes w\chi)(g) \circ I(w,\chi) = I(w,\chi) \circ \mathrm{Ind}_P^G(\sigma \otimes \chi)(g)$
- $I(w_1, w_2\chi) \circ I(w_2,\chi) = I(w_1 w_2, \chi)$
- $\mathrm{span}\{I(w,\chi) \mid w \in W_{\sigma,\chi}\} = \mathrm{End}_G(\mathrm{Ind}_P^G(\sigma \otimes \chi))$ for each $\chi \in \mathfrak{a}_P^*$

Intertwining operators on the C^*-algebra

Fix P and $\sigma \in \widehat{M}_{L^2} \rightsquigarrow \pi_{P,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$

$I(w,\chi) \in \mathrm{U}(\mathrm{Ind}_P^G H_\sigma)$ unitary intertwiners

Define: for each $w \in W_\sigma$:

- $u_w \in C(\mathfrak{a}_P^*, \mathrm{U}(\mathrm{Ind}_P^G H_\sigma))$ by $u_w(\chi) = I(w, w^{-1}\chi)$
- $\alpha_w \in \mathrm{Aut}\left(C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))\right)$ by $\alpha_w(f)(\chi) = f(w^{-1}\chi)$
- $\beta_w \in \mathrm{Aut}\left(C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))\right)$ by $\beta_w(f) = u_w \alpha_w(f) u_w^*$

So: $\quad \beta_w(f) : \chi \mapsto I(w, w^{-1}\chi) \circ f(w^{-1}\chi) \circ I(w, w^{-1}\chi)^*$

Define $C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$ to be the β-invariant subalgebra.

Theorem: $\pi_{P,\sigma}(C_r^*(G)) = C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$

$\subseteq$: intertwining property of $I(w,\chi)$. $\supseteq$: we'll use Stone-Weierstrass

Representations of $C_0(X, \mathsf{K}(H))^W$

Given:

X : locally compact Hausdorff $\qquad$ H : Hilbert space

W : finite group acting on X $\rightsquigarrow$ on functions: $\alpha_w(f) := f \circ w^{-1}$

$u : W \to C(X, \mathsf{U}(H))$ with $u_{w_1}\alpha_{w_1}(u_{w_2}) = u_{w_1 w_2}$

Define:

$\beta : W \to \mathrm{Aut}\left(C_0(X, \mathsf{K}(H))\right) : \quad \beta_w(f) = u_w \alpha_w(f) u_w^*$

$C_0(X, \mathsf{K}(H))^W$: the β-fixed subalgebra

Goal: Compute the irreps of the C^*-algebra $C_0(X, \mathsf{K}(H))^W$.

Representations of $C_0(X, \mathrm{K}(H))^W$, continued

Goal: Compute the irreps of the C^*-algebra $C_0(X, \mathrm{K}(H))^W$.

- Every irrep of $C_0(X, \mathrm{K}(H))^W$ occurs in an irrep of $C_0(X, \mathrm{K}(H))$
- Irreps of the latter are $\mathrm{eval}_x : C_0(X, \mathrm{K}(H)) \to \mathrm{K}(H)$
- $\mathrm{eval}_x(C_0(X, \mathrm{K}(H))^W) = \mathrm{K}_{W_x}(H)$, compact operators that commute with the unitary representation $w \mapsto u_w(x)$ of the isotropy group W_x.
- $\mathrm{K}_{W_x}(H) = \bigoplus_{\rho \in \widehat{W_x}} \mathrm{K}_{W_x}(H_\rho \otimes H'_\rho) \cong \bigoplus_{\rho \in \widehat{W_x}} \mathrm{K}(H'_\rho)$

Lemma: As x ranges over X/W and ρ ranges over $\widehat{W_x}$ the maps

$$C_0(X, \mathrm{K}(H))^W \xrightarrow{\mathrm{ev}_x} \mathrm{K}_{W_x}(H) \xrightarrow{\text{compress}} \mathrm{K}_{W_x}(H_\rho \otimes H'_\rho) \cong \mathrm{K}(H'_\rho)$$

are a complete set of inequivalent irreps of $C_0(X, \mathrm{K}(H))^W$. □

Representations of $C_0(\mathfrak{a}_P^*, \mathrm{K}(\operatorname{Ind}_P^G H_\sigma))^{W_\sigma}$

Lemma: As x ranges over X/W and ρ ranges over $\widehat{W_x}$ the maps

$$C_0(X, \mathrm{K}(H))^W \xrightarrow{\mathrm{ev}_x} \mathrm{K}_{W_x}(H) \xrightarrow{\text{compress}} \mathrm{K}_{W_x}(H_\rho \otimes H'_\rho) \cong \mathrm{K}(H'_\rho)$$

are a complete set of inequivalent irreps of $C_0(X, \mathrm{K}(H))^W$. □

Theorem: As χ ranges over $\mathfrak{a}_P^*/W_\sigma$, and L_τ ranges over the set of irreducible subrepresentations of $\operatorname{Ind}_P^G(\sigma \otimes \chi)$, the maps

$$C_0(\mathfrak{a}_P^*, \mathrm{K}(\operatorname{Ind}_P^G H_\sigma))^{W_\sigma} \xrightarrow{\mathrm{eval}_\chi} \mathrm{K}_{W_{\sigma,\chi}}(\operatorname{Ind}_P^G H_\sigma) \xrightarrow{\text{compress}} \mathrm{K}(L_\tau)$$

are a complete set of inequivalent irreps of $C_0(\mathfrak{a}_P^*, \mathrm{K}(\operatorname{Ind}_P^G H_\sigma))^{W_\sigma}$.

Proof: Recall $\operatorname{span}\{I(w,\chi) \mid w \in W_{\sigma,\chi}\} = \operatorname{End}_G(\operatorname{Ind}_P^G(\sigma \otimes \chi))$.
So the $W_{\sigma,\chi}$-isotypical components in $\operatorname{Ind}_P^G H_\sigma$ are precisely the G-isotypical components of $\operatorname{Ind}_P^G(\sigma \otimes \chi)$. □

$C_r^*(G)$ up to isomorphism

Recall: $P = M_P A_P N_P$ parabolic subgroup of G; $\sigma \in \widehat{M}_{L^2}$

$\rightsquigarrow \pi_{P,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_P^*, \mathrm{K}(\operatorname{Ind}_P^G H_\sigma))^{W_\sigma}$

Theorem: $\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\operatorname{Ind}_P^G H_\sigma))^{W_\sigma}$ is an isomorphism of C^*-algebras.

$[P,\sigma]$ ranges over the set of association classes of pairs (P,σ) (conjugacy of M_P and σ).

Proof: Three main parts. Each part relies on a big theorem from representation theory.

Part 1: The map goes into the direct sum. We use:

Theorem [Harish-Chandra]: For each P, each irrep of $K \cap M_P$ occurs in only finitely many σ.

Then by Frobenius reciprocity, each irrep of K occurs in only finitely many $\operatorname{Ind}_P^G H_\sigma$. So K-finite elements of $C_r^*(G)$ map into the direct sum, and K-finite elements are dense. □

$C_r^*(G)$ up to isomorphism, continued

Theorem: $\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$ is an isomorphism of C^*-algebras.

Part 2: The map is injective. We use:

Theorem [Langlands, Trombi]: Each $\pi \in \widehat{G}_{\mathrm{temp}}$ occurs in some $\mathrm{Ind}_P^G(\sigma \otimes \chi)$.

The kernel of $\bigoplus \pi_{P,\sigma}$ is therefore the intersection of the kernels of all of the irreducible representations of $C_r^*(G)$. But irreducible representations separate points in a C^*-algebra. □

$C_r^*(G)$ up to isomorphism, concluded

Theorem: $\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$ is an isomorphism of C^*-algebras.

Part 3: The map is surjective. We use:

Theorem [Langlands]: The only coincidences between irreducible subreps of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$s are the ones coming from conjugacy.

Recall: irreps of $\bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$

$$= \bigsqcup_{[P,\sigma]} \bigsqcup_{\chi \in \mathfrak{a}_P^*/W_\sigma} \big\{\text{irreducible } G\text{-subreps of } \mathrm{Ind}_P^G(\sigma \otimes \chi)\big\}$$

Restricted to the image of $C_r^*(G)$, those representations remain irreducible (obviously) and inequivalent (Langlands).

Everything in sight is liminal, so Kaplansky's Stone-Weierstrass theorem applies. □

Morita equivalence for C^*-algebras

A, B : C^*-algebras. What should $A \underset{\text{Morita}}{\sim} B$ mean?

(1) Morita equivalent as rings (if they both have 1)?

(2) Equivalent categories of Hilbert-space representations?

(3) Equivalent categories of Hilbert modules?

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(2) is a measure-theoretic notion rather than a topological one: eg $C([0,1]) \underset{(2)}{\sim} C(S^1)$.

(An interesting theorem [Beer]: A is type I $\iff A \underset{(2)}{\sim} C_0(X)$.)

(1) (in the unital case) and (3) (+ some conditions) turn out to be the right notion for most purposes.

The definition was originally formulated in a different way; see [Beer, 1982] and [Blecher, 1997] for the equivalence with (1) and (3).

Morita equivalence for C^*-algebras, continued

Definition [Rieffel, 1974]: A Morita equivalence between A and B is a bimodule ${}_AM_B$ with inner products ${}_A\langle\ |\ \rangle$ and $\langle\ |\ \rangle_B$ (values in A and B), satisfying:

- all of the C^*-module axioms (see Peter's first lecture)
- ${}_A\langle mb \mid n\rangle = {}_A\langle m \mid nb^*\rangle$ and $\langle am \mid n\rangle_B = \langle m \mid a^*n\rangle_B$
- $\overline{\mathrm{span}}_A\langle M \mid M\rangle = A$ and $\overline{\mathrm{span}}\langle M \mid M\rangle_B = B$
- ${}_A\langle l \mid m\rangle n = l\langle m \mid n\rangle_B$

Examples: • $\mathrm{K}(H) \underset{\mathrm{M}}{\sim} \mathbb{C}$ via the bimodule ${}_{\mathrm{K}}H_{\mathbb{C}}$ with $\langle\ |\ \rangle_{\mathbb{C}}$ as given, and ${}_{\mathrm{K}}\langle h \mid k\rangle = |h\rangle\langle k| : l \mapsto h\langle k \mid l\rangle$.

- $C_0(X, \mathrm{K}(H)) \underset{\mathrm{M}}{\sim} C_0(X)$ via the bimodule $C_0(X, H)$

Properties: $A \underset{\mathrm{M}}{\sim} B \Longrightarrow \widehat{A} \cong \widehat{B}$, $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$,
$\mathrm{Rep}(A) \cong \mathrm{Rep}(B)$, $K_*(A) \cong K_*(B)$ ...

Back to reductive groups

G : a real reductive group (eg GL_n or SL_n over $\mathbb{R}$ or $\mathbb{C}$). Last time:

Theorem: $C^*_r(G) \cong \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}^*_P, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$

— $P = M_P A_P N_P$: standard parabolic subgroup; σ an irreducible square-integrable unitary rep of M_P

— $\mathfrak{a}^*_P$ is a real vector space, and W_σ is a finite group of linear automorphisms

— $\mathrm{Ind}_P^G H_\sigma$ is the Hilbert space (compact picture) underlying the family of unitary reps $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ for $\chi \in \mathfrak{a}^*_P \cong \widehat{A_P}$

— W_σ acts on $C_0(\mathfrak{a}^*_P, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$ via $\beta_w(f) = u_w \alpha_w(f) u_w^*$ where

∘ $\alpha_w(f)(\chi) = f(w^{-1}\chi)$, and

∘ $u_w(\chi) = I(w, w^{-1}\chi)$ (Knapp-Stein normalised intertwiners)

Our next goal

We know:

Theorem: $C_r^*(G) \cong \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$

Here's where we are heading:

Theorem [Wassermann]: $C_r^*(G) \underset{\mathrm{M}}{\sim} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma$

$\underset{\mathrm{M}}{\sim}$ commutes with $\bigoplus$. So we fix one (P, σ) and aim to show

$$\boxed{C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma} \underset{\mathrm{M}}{\sim} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma}$$

A thought experiment

Imagine that $I(w, \chi)$ is a scalar, for every $w \in W_\sigma$ and $\chi \in \mathfrak{a}_P^*$.

(this almost never happens)

Then we have $\beta_w(f) = u_w \alpha_w(f) u_w^* = \alpha_w(f)$, so

$$C_0\big(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)\big)^{W_\sigma} \cong C_0(\mathfrak{a}_P^*/W_\sigma, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)) \underset{\mathrm{M}}{\sim} C_0(\mathfrak{a}_P^*/W_\sigma)$$

(this sometimes happens)

Moral: To understand $C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$ we need to understand scalar/non-scalar intertwiners

The R-group

Theorem [Knapp-Stein]: Set $W'_{\sigma,\chi} = \{w \in W_{\sigma,\chi} \mid I(w,\chi) \text{ scalar}\}$.

(1) $W_{\sigma,\chi} = W'_{\sigma,\chi} \rtimes R_{\sigma,\chi}$ for some subgroup $R_{\sigma,\chi} \cong (\mathbb{Z}/2\mathbb{Z})^n$

(2) $\{I(r,\chi) \mid r \in R_{\sigma,\chi}\}$ is linearly independent

(3) $W'_{\sigma,\chi} \subseteq W'_{\sigma}$, and $R_{\sigma,\chi} \hookrightarrow W_\sigma \twoheadrightarrow R_\sigma$ is injective.

(4) The $I(w,\chi)$ can (and will) be normalised so that $I(w,\chi) = 1$ for all χ and all $w \in W'_{\sigma,\chi}$.

Notation: $q_\chi : W_{\sigma,\chi} \xrightarrow{\text{quotient}} R_{\sigma,\chi}, \quad \lambda_\chi : R_{\sigma,\chi} \xrightarrow{\text{regular}} \mathrm{U}(\ell^2 R_{\sigma,\chi})$

Corollary: The reps $w \mapsto I(w,\chi)$ and $w \mapsto \lambda_0 \circ q_0(w)$ of $W_{\sigma,\chi}$ are <u>quasi-equivalent</u> (same irreducible constituents).

Proof: $I(\cdot,\chi) \underset{\text{quasi}}{\overset{(2),(4)}{\sim}} (\lambda_\chi \circ q_\chi) \underset{\text{quasi}}{\overset{(3)}{\sim}} (\lambda_0 \circ q_0)$. □

Morita equivalence for $C_0(X, \mathrm{K}(H))^W$

Given:

X : locally compact Hausdorff $\quad$ H, L : Hilbert spaces

W : finite group acting on X $\rightsquigarrow$ on functions: $\alpha_w(f) := f \circ w^{-1}$

$u : W \to C(X, \mathrm{U}(H))$ and $v : W \to C(X, \mathrm{U}(L))$

$u_{w_1}\alpha_{w_1}(u_{w_2}) = u_{w_1 w_2}$ and $v_{w_1}\alpha_{w_1}(v_{w_2}) = v_{w_1 w_2}$

Define:

$\beta : W \to \mathrm{Aut}\left(C_0(X, \mathrm{K}(H))\right)$: $\beta_w(f) = u_w \alpha_w(f) u_w^*$

$\gamma : W \to \mathrm{Aut}\left(C_0(X, \mathrm{K}(L))\right)$: $\gamma_w(f) = v_w \alpha_w(f) v_w^*$

Goal: when is $C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(X, \mathrm{K}(L))^W$?

Morita equivalence for $C_0(X, \mathrm{K}(H))^W$, continued

For each $x \in X$, $u(x) : W \to \mathrm{U}(H)$ and $v(x) : W \to \mathrm{U}(L)$ are unitary representations of the isotropy group W_x.

Theorem: If $u(x)$ and $v(x)$ are quasi-equivalent for each $x \in X$ then $C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(X, \mathrm{K}(L))^W$.

Proof: set $A = C_0(X, \mathrm{K}(L))^W$, $B = C_0(X, \mathrm{K}(H))^W$, and $M = C_0(X, \mathrm{K}(H, L))^W$ where $w(f) = v_w \alpha_w(f) u_w^*$

- M is an A-B bimodule (pointwise composition)
- Inner products: ${}_A\langle f_1 \mid f_2 \rangle = f_1 f_2^*$, $\langle f_1 \mid f_2 \rangle_B = f_1^* f_2$.
- Most conditions are easy to check (embed everything into the C^*-algebra $C_0(X, \mathrm{K}(H \oplus L))$).

The work: $\overline{\mathrm{span}}\langle M \mid M \rangle_B = B$ and $\overline{\mathrm{span}}_A\langle M \mid M \rangle = A$

Morita equivalence for $C_0(X, \mathsf{K}(H))^W$, continued

We're left to show:

$$(??) \quad C_0(X, \mathsf{K}(H))^W = \overline{\operatorname{span}} \left\{ f_1^* f_2 \mid f_1, f_2 \in C_0(X, \mathsf{K}(H, L))^W \right\}$$

Recall Kaplansky's Stone-Weierstrass theorem: LHS=RHS if

(1) irreps of LHS remain irreducible on RHS; and

(2) inequivalent irreps of LHS remain inequivalent on RHS

Recall irreps of LHS factor through pointwise evaluations

$$C_0(X, \mathsf{K}(H))^W \longrightarrow \mathsf{K}_{W_x}(H)$$

with equivalences arising only if $Wx = Wx'$.

Morita equivalence for $C_0(X, \mathsf{K}(H))^W$, concluded

We're left to show:

$$(??) \quad C_0(X, \mathsf{K}(H))^W = \overline{\operatorname{span}} \left\{ f_1^* f_2 \mid f_1, f_2 \in C_0(X, \mathsf{K}(H, L))^W \right\}$$

(1) irreps of LHS remain irreducible on RHS; and

(2) inequivalent irreps of LHS remain inequivalent on RHS

— Evaluating (??) at x gives:

$$\mathsf{K}_{W_x}(H) = \overline{\operatorname{span}} \{ k_1^* k_2 \mid k_1, k_2 \in \mathsf{K}_{W_x}(H, L) \}$$

which is true because H and L are quasi-equivalent over W_x.

— So (1) holds, and (2) holds for irreps sitting over the same x.

— RHS of (??) is a $C_0(X/W)$-submodule, so (2) holds for irreps sitting over different x. □

Back to $C_r^*(G)$

Return to $C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$, where W_σ acts via

$$u_w(\chi) = I(w, w^{-1}\chi) \qquad (w \in W_\sigma,\ \chi \in \mathfrak{a}_P^*)$$

Recall: $W_\sigma = W'_\sigma \rtimes R_\sigma, \quad q : W_\sigma \xrightarrow{\text{quot}} R_\sigma, \quad \lambda : R_\sigma \xrightarrow{\text{reg}} \mathrm{U}(\ell^2 R_\sigma)$

Recall [Knapp-Stein]: $u(\chi) \underset{\text{quasi}}{\sim} (\lambda \circ q)\big|_{W_{\sigma,\chi}}$ for every χ.

Corollary: $C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma} \underset{\mathrm{M}}{\sim} C_0(\mathfrak{a}_P^*, \mathrm{K}(\ell^2 R_\sigma))^{W_\sigma}$

$$\underset{\mathrm{M}}{\sim} C_0(\mathfrak{a}_P^*/W'_\sigma, \mathrm{K}(\ell^2 R_\sigma))^{R_\sigma}$$

where R_σ acts by $r(f) = \lambda(r)\alpha_r(f)\lambda(r)^*$.

The last piece

X : locally compact Hausdorff, R : finite group acting on X

Let R act on $C_0(X, \mathrm{K}(\ell^2 R))$ by $r(f) = \lambda(r)\alpha_r(f)\lambda(r)^*$.

Lemma: $C_0(X, \mathrm{K}(\ell^2 R))^R \cong C_0(X) \rtimes R$

Proof: Recall that $C_0(X) \rtimes R = \left\{\sum_{r\in R} f_r r \mid f_r \in C_0(X)\right\}$ with $rf = \alpha_r(f)r$ and $r^* = r^{-1}$.

Let $\{\varepsilon_s \mid s \in R\}$ be the standard basis for $\ell^2 R$.

Define $\Phi : C_0(X) \rtimes R \to C_0(X, \mathrm{K}(\ell^2 R))^R$ by

$$\Phi(fr)(x) : \varepsilon_s \mapsto f(rs^{-1}x)\varepsilon_{sr^{-1}}$$

and $\Psi : C_0(X, \mathrm{K}(\ell^2 R))^R \to C_0(X) \rtimes R$ by

$$\Psi(f) = \sum_{r\in R} \langle \varepsilon_1 \mid f(\ )\varepsilon_r \rangle r.$$

They're inverse isomorphisms of C^*-algebras. □

The end result

Theorem [A. Wassermann]: $C_r^*(G) \underset{M}{\sim} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma$

Example: G complex semisimple: take $L = MA$ minimal

$$C_r^*(G) \underset{M}{\sim} \bigoplus_{\sigma \in \widehat{M}/W} C_0(\mathfrak{a}^*/W_\sigma) = C_0(\widehat{L}/W) = C_0(\widehat{G}_{\mathrm{temp}})$$

Example: $G = \mathrm{SL}_2(\mathbb{R})$:

$$C_r^*(\mathrm{SL}_2(\mathbb{R})) \underset{M}{\sim} C_0(\mathbb{R}/\{\pm 1\}) \oplus (C_0(\mathbb{R}) \rtimes \{\pm 1\}) \oplus \bigoplus_{\text{countable}} \mathbb{C}$$

p-adic groups? [Plymen et al], [Afgoustidis-Aubert]

— non-trivial cocycles from choices of isomorphisms $w\sigma \cong \sigma$
$\rightsquigarrow$ twisted crossed products

— you don't always have $W'_{\sigma,\chi} \subseteq W'_\sigma$ and $R_{\sigma,\chi} \hookrightarrow R_\sigma$ ... but sometimes you do, and then the same arguments apply

$\widehat{G}_{\text{temp}}$ as a topological space

For each $[P, \sigma]$ and each $\chi \in \mathfrak{a}_P^*$ let $R_\sigma^\chi = \{r \in R_\sigma \mid r\chi \in W'_\sigma \chi\}$.

Define equivalence on $(\mathfrak{a}_P^*/W_\sigma) \times \widehat{R_\sigma}$ by

$$(W_\sigma\chi, \rho) \sim (W_\sigma\chi', \rho') \iff W_\sigma\chi = W_\sigma\chi' \text{ and } \rho\big|_{R_\sigma^\chi} = \rho'\big|_{R_\sigma^{\chi'}}$$

Now a general theorem of [Williams, 1981] gives:

Corollary: $\widehat{G}_r \cong \bigsqcup_{[P,\sigma]} \Big((\mathfrak{a}_P^*/W_\sigma) \times \widehat{R_\sigma} \Big) / \sim$

See [Echterhoff, 2017], [Echterhoff-Emerson, 2011] for more about the duals of crossed products.

Example: $\mathfrak{a}_P^*/W_\sigma \to \widehat{G}_r$, $W_\sigma\chi \mapsto (W_\sigma\chi, \text{triv}_{R_\sigma})$ embeds $\mathfrak{a}_P^*/W_\sigma$ as a dense open subset of the $[P, \sigma]$-component of $\widehat{G}_r$.

Coda: parabolic induction, C^*-algebraically

G real reductive, $P = MAN$ parabolic.

Theorem [Clare]: There is a C^*-module ${}_{C_r^*(G)}C_r^*(G/N)^{\langle\ |\ \rangle}_{C_r^*(L)}$ with

$$\operatorname{Ind}_P^G H \cong C_r^*(G/N) \otimes_{C_r^*(L)} H$$

for all tempered unitary reps H of L. Moreover, $C_r^*(G)$ acts on $C_r^*(G/N)$ by 'compact' operators (in the Hilbert C^*-module sense).

Idea: for finite groups $\operatorname{ind}_H^G V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$, which clarifies some properties of induction (Frobenius reciprocity, Mackey's induction-restriction formula, etc).

Does $C_r^*(G/N)$ do the same for Ind_P^G?

Tempered parabolic restriction

$M_B^{\langle\,|\,\rangle}$, $A \to \mathrm{K}(M)$ $\rightsquigarrow$ Frobenius reciprocity for operator modules

...and in special cases also for Hilbert space representations:

Theorem [Clare-Crisp-Higson]: There is a $C_r^*(G)$-valued inner product on $C_r^*(G/N)$.

Proof: $\mathrm{K}(C_r^*(G/N))^{W_P} \cong \bigoplus_{[Q,\sigma]} C_0(\mathfrak{a}_Q^*, \mathrm{K}(\mathrm{Ind}_Q^G H_\sigma))^{W_\sigma}$ (over $Q \subseteq L$) is a direct-summand of $C_r^*(G)$, and we use the inner product $\langle f_1 \mid f_2\rangle = \frac{1}{|W|}\sum_w w\,(|f_1\rangle\langle f_2|)$. □

This gives a functor $\mathrm{Res}_P^G : \mathrm{U}_{\mathrm{temp}}(G) \to \mathrm{U}_{\mathrm{temp}}(L)$ adjoint to Ind_P^G.

Example: $G = \mathrm{SL}_2(\mathbb{R})$, $P =$ upper-triangular, $\tau \in \widehat{L}$:

$$\mathrm{Res}_P^G \,\mathrm{Ind}_P^G(\tau) = \begin{cases} \tau & \text{if } \tau = \mathrm{triv}_L \\ \tau \oplus \tau^{-1} & \text{otherwise.} \end{cases}$$

Tempered parabolic restriction

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Thanks for having me!