The Mackey-Rieffel-Green mashine

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AIM of this lecture

Reminder: If $\alpha : G \to \mathrm{Aut}(A)$ is an action, we defined the maximal and reduced crossed products $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,\mathrm{red}} G$ as completions of $C_c(G, A)$ with respect to certain C^*-norms.

AIM: Describe (if possible) the spaces

$$(A \rtimes_\alpha G)\widehat{} \quad \text{and} \quad \mathrm{Prim}(A \rtimes_\alpha G) \quad \text{via}$$

(1) the spaces $\widehat{A}$ resp. $\mathrm{Prim}(A)$.

(2) the action $G \curvearrowright \mathrm{Prim}(A), (g, P) \mapsto g \cdot P := \alpha_g(P)$.

(3) The representation theory of $A_P \rtimes G_P$, $P \in \mathrm{Prim}(A)$, with $G_P := \{g \in G : g \cdot P = P\}$ the stabilizerof P in G and A_P the simple subquotient of Acorreponding to P.

This will generalize Mackey's theory for group extensions.

Convention: In this lecture we will ignore all modular functions!

Group extensions

Let $G := N \rtimes H$ be a semi-direct product group. We get an action

$$\alpha : H \to \mathrm{Aut}(C^*(N)); \quad \alpha_h(\varphi)(n) := \varphi(h^{-1} \cdot n) \quad \forall \varphi \in C_c(N).$$

If $V : N \rtimes H \to UM(B)$ is a unitary representation of G, let $\widetilde{V|_N} : C^*(N) \to \mathcal{B}(H)$ be the integrated form of $V|_N$. Then $(\widetilde{V|_N}, V|_H)$ is a covariant representation of $(C^*(N), H, \alpha)$.

We get an isomorphism $C^*(N \rtimes H) \cong C^*(N) \rtimes_\alpha H$ via

$$\Phi : C_c(N \rtimes H) \to C_c(H, C_c(N)); \quad \Phi(f)(h)(n) := f(n, h).$$

More generally: if $N \lhd G$, then there exist an action of $\dot{G} := G/N$ on $C^*(N) \otimes \mathcal{K}(H)$ such that

$$C^*(G) \otimes \mathcal{K}(H) \cong (C^*(N) \otimes \mathcal{K}(H)) \rtimes_\alpha \dot{G}.$$

(or use Phil Green's theory of twisted crossed products instead.)

Hilbert C^*-modules

Recall that a Hilbert A-module is a right Banach A-module $\mathcal{X}$ together with an A-valued inner product

$$\langle \cdot, \cdot \rangle_A : \mathcal{X} \times \mathcal{X} \to A$$

such that $\langle \xi, \xi \rangle_A > 0$ iff $\xi \neq 0$ and $\forall \xi, \eta \in \mathcal{X}, a \in A$:

$$\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^* \quad \text{and} \quad \langle \xi, \eta \rangle_A a = \langle \xi, \eta a \rangle_A$$

We say that $\mathcal{X}$ is full if $\overline{\text{span}}\{\langle \xi, \eta \rangle_A : \xi, \eta \in \mathcal{X}\} = A$.

The algebra of adjointable operators

$$\mathcal{L}_A(\mathcal{X}) := \{T : \mathcal{X} \to \mathcal{X} : \exists\, T^* : \mathcal{X} \to \mathcal{X} \text{ s.t } \langle T\xi, \eta \rangle_A = \langle \xi, T^*\eta \rangle_A\}.$$

becomes a C^*-algebra w.r.t. operator norm. Recall also the ideal of compact operators

$$\mathcal{K}(\mathcal{X}) = \overline{\text{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in \mathcal{X}\} \quad \Theta_{\xi,\eta}(\zeta) = \xi \cdot \langle \eta, \zeta \rangle_A.$$

Morita equivalence

Definition Two C^*-algebras A, B are called **Morita equivalent**, if there exists a **full** Hilbert B-module $\mathcal{X}$ and an isomorphism $\Phi : A \xrightarrow{\cong} \mathcal{K}(\mathcal{X})$. We call $(\mathcal{X}, \Phi)$ an A-B-**equivalence bimodule**.

Notice: We obtain a left A-valued inner product on $\mathcal{X}$ by ${}_A\langle \xi, \eta \rangle := \Phi^{-1}(\Theta_{\xi,\eta})$. This satisfies the compatibility relation

$$ {}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B \quad \forall \xi, \eta, \zeta \in \mathcal{X}. $$

Moreover, we get $B \cong \mathcal{K}({}_A\mathcal{X})$!

There is an **inverse** B-A-equivalence bimodule $(\mathcal{X}^*, \Phi^*)$ with $\mathcal{X}^* = \{\xi^* : \xi \in \mathcal{X}\}$ with inner products $\langle \xi^*, \eta^* \rangle_A := {}_A\langle \xi, \eta \rangle$ and ${}_B\langle \xi^*, \eta^* \rangle := \langle \xi, \eta \rangle_B$ and actions

$$ \Phi^*(b)\xi^* := (\xi b^*)^*, \quad (\xi^* a) := (\Phi(a^*)\xi)^* $$

All these operations are encoded in the **Linking-algebra**

$$ L(\mathcal{X}) := \begin{pmatrix} A & \mathcal{X} \\ \mathcal{X}^* & B \end{pmatrix}. $$

Examples for Morita equivalences

1. If $\mathcal{X}$ is any full Hilbert B-module, then $(\mathcal{X}, \mathrm{id}_{\mathcal{K}})$ is a $\mathcal{K}(\mathcal{X})$-B equivalence bimodule.
2. Every Hilbert space H gives the $\mathcal{K}(H)$-$\mathbb{C}$ equivalence bimodule $(H, \mathrm{id}_{\mathcal{K}})$.
3. If A is a C^*-algebra, then (A, id_A) becomes an A-A equivalence bimodule w.r.t $\langle a, b\rangle_A = a^* b$.
4. Combining (2) and (3) we get an $A \otimes \mathcal{K}(H)$-A equivalence bimodule $A \otimes H$ w.r.t.

$$\langle a \otimes \xi, b \otimes \eta\rangle_A = a^* b\langle \xi, \eta\rangle_{\mathbb{C}}$$

5. More interesting examples will follow below!

Morita equivalences preserve: the spaces $\mathrm{Rep}(A)$, $\widehat{A}, \mathrm{Prim}(A)$, nuclearity, simplicity, type I, CCR, continuous trace, K-theory, etc.

Exceptions: unitality, commutativity!

The Morita category

The Morita category $\mathcal{M}\mathrm{orC}^*$ is the category with

1. C^*-algebras as objects, and
2. $\mathrm{Mor}(A, B)$ consisting of equivalence classes of pairs $(\mathcal{X}, \Phi)$ with $\mathcal{X}$ a Hilbert B-module and $\Phi : A \to \mathcal{L}_B(\mathcal{X})$ a $*$-hom..

Here

$$(\mathcal{X}_1, \Phi_1) \sim (\mathcal{X}_2, \Phi_2) \Longleftrightarrow \Phi_1(A)\mathcal{X}_1 \cong \Phi_2(A)\mathcal{X}_2.$$

Composition of $(\mathcal{X}_1, \Phi_1) \in \mathrm{Mor}(A, B)$ with $(\mathcal{X}_2, \Phi_2) \in \mathrm{Mor}(B, C)$ is defined via

$$(\mathcal{X}_2, \Phi_2) \circ (\mathcal{X}_1, \Phi_1) = (\mathcal{X}_1 \otimes_B \mathcal{X}_2, \Phi_1 \otimes 1) \in \mathrm{Mor}(A, C).$$

Here $\mathcal{X}_1 \otimes_B \mathcal{X}_2$ is the Hausdorff completion of $\mathcal{X}_1 \odot \mathcal{X}_2$ w.r.t

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_C = \langle \eta_1, \Phi(\langle \xi_1, \xi_2 \rangle_B)\eta_2 \rangle_C.$$

The Morita category

1. The identity morphisms id_A of A is represented by (A, id_A), since $A \otimes_A \mathcal{X} \cong \Phi(A)\mathcal{X}$ for every $(\mathcal{X}, \Phi) \in \mathrm{Mor}(A, B)$.
2. The isomorphisms in $\mathrm{Mor}(A, B)$ are the A-B Morita equivalences $(\mathcal{X}, \Phi)$ with inverse $(\mathcal{X}^*, \Phi^*)$: We have

$$\mathcal{X}^* \otimes_A \mathcal{X} \cong B \quad \text{via} \quad \xi^* \otimes \eta \mapsto \langle \xi, \eta \rangle_B$$
$$\mathcal{X} \otimes_B \mathcal{X}^* \cong A \quad \text{via} \quad \xi \otimes \eta^* \mapsto {}_A\langle \xi, \eta \rangle.$$

3. For every C^*-algebra B we have $\mathrm{Rep}(B) = \mathrm{Mor}(B, \mathbb{C})$. Thus composition with an element on $(\mathcal{X}, \Phi) \in \mathrm{Mor}(A, B)$ induces

$$\mathrm{Ind}^{\mathcal{X}} : \mathrm{Rep}(B) \to \mathrm{Rep}(A); (H, \pi) \mapsto (\mathcal{X} \otimes_B H, \Phi \otimes 1).$$

   It is very easy to check that this preserves weak containment.
4. Composition with a Morita equivalence induces homeomorphisms $\widehat{A} \cong \widehat{B}$ (and similarly $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$.)

Mackey induction

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action, $H < G$ a closed subgroup.

Define a $C_c(H, A)$-valued inner product on $C_c(G, A)$ via

$$\langle \xi, \eta \rangle_{C_c(H,A)} = (\xi^* * \eta)|_H$$

and right action of $\varphi \in C_c(H, A)$ on $\xi \in \mathcal{X}_0$ by

$$\xi \cdot \varphi(s) := \xi|_{sH} * \varphi = \int_H \xi(sh)\alpha_{sh}(\varphi(h^{-1}))\, dh.$$

Then $\mathcal{X}_0$ completes to a Hilbert $A \rtimes_\alpha H$-module $\mathcal{X}$ and we have a left action

$$\Phi : A \rtimes_\alpha G \to \mathcal{L}(\mathcal{X}); \Phi(f)\xi = f * \xi \quad f, \xi \in C_c(G, A).$$

Composition with $(\mathcal{X}, \Phi)$ in $\mathcal{M}\mathrm{orC}^*$ gives Rieffel's version of the Mackey induction

$$\mathrm{Ind}_H^G : \mathrm{Rep}(A \rtimes H) \to \mathrm{Rep}(A \rtimes G).$$

It automatically preserves weak containment!

The imprimitivity theorem

Let $\mathcal{X} = \overline{C_c(G, A)}$ be the Hilbert $A \rtimes H$-module of the previous slide. Then there is an action

$$M : C_0(G/H) \to \mathcal{L}(\mathcal{X}); (M(f)\xi)(s) = f(sH)\xi(s)$$

Together with the convolution action of $C_c(G, A)$ this combines to a left action

$$\Phi : (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau} G \overset{\cong}{\to} \mathcal{K}(\mathcal{X}).$$

This gives Phil Green's imprimitivity theorem '78:

$$(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau} G \sim_M A \rtimes_\alpha H.$$

Corollary: Mackey's imprimitivity theorem

For a cov. rep. $\pi \rtimes U \in \mathrm{Rep}(A \rtimes_\alpha G)$ the following are equivalent

1. $\pi \rtimes U$ is induced from some $\rho \rtimes V \in \mathrm{Rep}(A \rtimes_\alpha H)$;
2. $\exists$ a nondeg. rep. $P : C_0(G/H) \to \mathcal{B}(H_\pi)$ s.t.

$$(\pi \otimes P) \rtimes U \in \mathrm{Rep}((A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau} G).$$

A (slight) generalization

Let $H < G$ and $\beta : H \to \mathrm{Aut}(B)$ given. Define the induced algebra

$$\mathrm{Ind}_H^G B := \left\{ F : G \to B : F \text{ cont. } \begin{Bmatrix} F(sh) = \beta_{h^{-1}}(F(s)) \\ (sH \mapsto \|F(s)\|) \in C_0(G/H) \end{Bmatrix} \right\}$$

with action $\big(\mathrm{Ind}\,\beta(s)F\big)(t) := F(s^{-1}t).$

Special case If $\alpha : G \to \mathrm{Aut}(A)$, then :

$$\mathrm{Ind}_H^G(A, \alpha|_H) \cong C_0(G/H, A) \quad \text{via} \quad F \mapsto \big[sH \mapsto \alpha_s(F(s))\big].$$

Then $\mathcal{X}_0 = C_c(G, B)$ completes to an $\mathrm{Ind}_H^G B \rtimes_{\mathrm{Ind}\,\alpha} G - B \rtimes_\alpha H$ equivalence bimodule $(\mathcal{X}, \Phi)$ with respect to

$$\langle \xi, \eta \rangle_{C_c(H,B)}(h) = \int_G \xi(t^{-1})^* \beta_h(\eta(t^{-1}h))\, dh$$

$$\xi \cdot \varphi(s) = \int_h \beta_h(\xi(sh)\varphi(h^{-1}))\, dh$$

$$\big(\Phi(f)\xi\big)(s) = \int_G f(t, s)\xi(t^{-1}s)\, dt$$

$\xi, \eta \in \mathcal{X}_0$, $\varphi \in C_c(H, B)$, $f \in C_c(G, \mathrm{Ind}_H^G B)$.

The generalized imprimitivity theorem

In particular, we obtain a homeomorphism

$$\operatorname{Ind}^{\mathcal{X}} : (B \rtimes_\beta H)^\wedge \to (\operatorname{Ind}_H^G B \rtimes_{\operatorname{Ind}\beta} G)^\wedge$$

and similarly $\quad \operatorname{Prim}(B \rtimes_\beta H) \cong \operatorname{Prim}(\operatorname{Ind}_H^G B \rtimes_{\operatorname{Ind}\beta} G)$.

Theorem E '90 For a system (A, G, α) TFAE

1. $(A, G, \alpha) \cong (\operatorname{Ind}_H^G B, G, \operatorname{Ind}\beta)$ for some (B, H, β);
2. $\exists$ a G-equivariant continuous map $\varphi : \operatorname{Prim}(A) \to G/H$.

Proof (2)⇒(1): If φ exists, let $J := \cap\{P : \varphi(P) = eH\}$ and $B := A/J$. Then use the Dauns-Hofmann theorem to check that

$$\Phi : A \overset{\cong}{\to} \operatorname{Ind}_H^G B; \quad \Phi(a)(s) := \alpha_s(a) + J$$

is an isomorphism. □

An application

Recall the Mautner group $\mathbb{C}^2 \rtimes \mathbb{R}$ with $x \cdot (z, w) = (e^{ix}z, e^{2\pi ix}w)$.
The action $\hat{\alpha}$ on $\widehat{\mathbb{C}^2} \cong \mathbb{C}^2$ is given by $x \cdot (z, w) = (e^{-ix}z, e^{-2\pi ix}w)$.
Hence

$$C^*(\mathbb{C}^2 \rtimes \mathbb{R}) \cong C^*(\mathbb{C}^2) \rtimes_\alpha \mathbb{R} \cong C_0(\mathbb{C}^2) \rtimes_{\hat{\alpha}} \mathbb{R}$$

There is a large invariant ideal $I := C_0((\mathbb{C} \setminus \{0\})^2)$ in $C_0(\mathbb{C}^2)$ which gives a large ideal

$$I \rtimes_{\hat{\alpha}} \mathbb{R} \subseteq C^*(\mathbb{C}^2 \rtimes \mathbb{R}).$$

Consider $\varphi : \widehat{I} := (\mathbb{C} \setminus \{0\})^2 \to \mathbb{T} \cong \mathbb{R}/\mathbb{Z}; \quad (z, w) \mapsto \frac{w}{|w|}$.

We get $I \cong \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}} C_0(\varphi^{-1}(\{1\})) = \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}} C_0(\mathbb{C}^* \times (0, \infty)))$ w.r.t.

$$\mathbb{Z} \curvearrowright \mathbb{C}^* \times (0, \infty) \cong \mathbb{T} \times (0, \infty)^2 \qquad n \cdot (z, s, t) = (e^{-in}z, s, t)$$

Hence
$I \rtimes_\alpha \mathbb{R} \sim_M C_0(\mathbb{T} \times (0, \infty)^2) \rtimes_\beta \mathbb{Z} \cong \big(C(\mathbb{T}) \rtimes_\beta \mathbb{Z}\big) \otimes C_0\big((0, \infty)^2\big)$.

Mackey's orbit method

Recall that a locally closed subset $E \subseteq \mathrm{Prim}(A)$ corresponds to a subquotient I/J with $J = \ker E$ $(= \cap_{P \in E} P)$, $I = \ker(\bar{E} \setminus E)$.

For $\alpha : G \to \mathrm{Aut}(A)$ let $G \curvearrowright \mathrm{Prim}(A); (s, P) \mapsto s \cdot P := \alpha_s(P)$.

We say this action is regular if the following hold

(1) all orbits $G(P) = \{s \cdot P : s \in G\}$ are locally closed in $\mathrm{Prim}(A)$.

(2) For all P, the map $G/G_P \to G(P);\ sG_P \mapsto s \cdot P$ is a homeomorphism (with $G_P := \{s \in G : s \cdot P = P\}$).

(3) A is separable, or $\mathrm{Prim}(A)/G$ is almost Hausdorff.

Item (1) implies that $\forall\ P \in \mathrm{Prim}(A)\ \exists$ a G-invariant subquotient $A_{G(P)} = I_{G(P)}/J_{G(P)}$ of A such that $G(P) \cong \mathrm{Prim}(A_{G(P)})$.

Item (2) together with the theorem on the previous slide gives

$$A_{G(P)} \cong_G \mathrm{Ind}_{G_P}^{G} A_P \quad \text{hence} \quad A_{G(P)} \rtimes_\alpha G \sim_M A_P \rtimes_{\alpha_P} G_P,$$

for $A_P := I_P/P$ the subquotient of A with $\{P\} = \mathrm{Prim}(A_P)$.

Item (3) implies that every primitive ideal (resp. irreducible rep.) of $A \rtimes_\alpha G$ belongs to exactly one subquotient $A_{G(P)} \rtimes_\alpha G$.

The orbit method

Indeed, if $0 \to I \to A \to A/I \to 0$ is a short exact sequence of G-algebras, we always get a short exact sequence

$$0 \to I \rtimes_\alpha G \to A \rtimes_\alpha G \to (A/I) \rtimes_\alpha G \to 0$$

of the full crossed products! (Not always true for $A \rtimes_{\text{red}} G$!)

In particular, G-invariant subquotients I/J of A correspond to subquotients $(I/J) \rtimes_\alpha G = (I \rtimes_\alpha G)/(J \rtimes_\alpha G)$ of $A \rtimes_\alpha G$.

Thus, under the assumptions of the previous slide we get

$$\mathrm{Prim}(A \rtimes_\alpha G) = \dot{\bigcup}_{G(P) \in \mathrm{Prim}(A)/G} \mathrm{Prim}(A_{G(P)} \rtimes_\alpha G)$$

and $\mathrm{Prim}(A_{G(P)} \rtimes_\alpha G)$ is homeomorphic to $\mathrm{Prim}(A_P \rtimes_\alpha G_P)$ via Mackey induction!

Similarly

$$(A \rtimes_\alpha G)\widehat{\ } = \dot{\bigcup}_{G(P) \in \mathrm{Prim}(A)/G} (A_{G(P)} \rtimes_\alpha G)\widehat{\ }$$

and $(A_{G(P)} \rtimes_\alpha G)\widehat{\ } \cong (A_P \rtimes_\alpha G_P)\widehat{\ }$ via Mackey induction!

Mackey's little group method

Question: How do A_P and $A_P \rtimes_\alpha G_P$ look like?

Special case 1:
If $A = C_0(X)$, then $X \cong \widehat{A}$ ($\cong \mathrm{Prim}(A)$) via $x \mapsto \epsilon_x$ (eval. at x).

Then $A_x = C(\{x\}) \cong \mathbb{C}$, and $A_x \rtimes G_x \cong C^*(G_x)$. We get

$$(C_0(X) \rtimes_\alpha G)\widehat{\ } = \dot{\bigcup}_{G(x)\in X/G} \{\mathrm{Ind}_{G_x}^G(\epsilon_x \rtimes \rho) : \rho \in \widehat{G_x}\}.$$

(and similarly for $\mathrm{Prim}(C_0(X) \rtimes_\alpha G)$).

An Example

Let $G = \mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$. Write $H := \mathrm{SL}_2(\mathbb{R})$. Then

$$C^*(G) \cong C^*(\mathbb{R}^2) \rtimes \mathrm{SL}_2(\mathbb{R}) \cong C_0(\mathbb{R}^2) \rtimes_\alpha SL_2(\mathbb{R})$$

with $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathbb{R}^2; (A, x) \mapsto A^t x$.

Then $\mathbb{R}^2/H = \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\} \dot\cup \{\mathbb{R}^2 \setminus \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\}\}$ with stabilizers

$$H_{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} = \mathrm{SL}_2(\mathbb{R}) \quad \text{and} \quad H_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = \{\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) : a \in \mathbb{R}\} \cong \mathbb{R}.$$

Thus

$$\widehat{G} = \widehat{\mathrm{SL}_2(\mathbb{R})} \dot\cup \left\{ \mathrm{Ind}_{\mathbb{R}^2 \rtimes \mathbb{R}}^{G} \chi_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \rtimes \mu : \mu \in \widehat{\mathbb{R}} \right\},$$

where $\chi_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = e^{2\pi i x}$ is the character of $\mathbb{R}^2$ corresponding to the vector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{R}^2 \cong \widehat{\mathbb{R}^2}$.

Mackey's little group method

Special case 2: Let $\alpha : G \to \mathrm{Aut}(A)$ with A type I. Then

$$\widehat{A} \cong \mathrm{Prim}(A) \quad \text{via} \quad \pi \mapsto \ker \pi \quad \text{and} \quad A_\pi = \mathcal{K}(H_\pi),$$

$$\text{Note:} \quad \mathrm{Aut}(\mathcal{K}(H_\pi)) = P\mathcal{U}(H_\pi) = \mathcal{U}(H_\pi)/\mathbb{T}.$$

Let $\alpha_\pi : G_\pi \to P\mathcal{U}(H_\pi) = \mathrm{Aut}(\mathcal{K}(H_\pi))$ be the action of G_π and let $c : P\mathcal{U} \to \mathcal{U}$ be a Borel section. Then $W := c \circ \alpha_\pi : G_\pi \to \mathcal{U}(H_\pi)$ satisfies $\mathrm{Ad}(W_s W_t) = \alpha_\pi(s)\alpha_\pi(t) = \alpha_\pi(st) = \mathrm{Ad}\, W_{st}$.

Thus there exist a Borel map $\omega_\pi : G_\pi \times G_\pi \to \mathbb{T}$ such that

$$W_s W_t = \omega_\pi(s,t) W_{st}.$$

On elements $s, t, r \in G_\pi$ one easily checks that $\omega_\pi \in Z^2(G_\pi, \mathbb{T})$.

Theorem (Mackey)

$$\mathcal{K}(H_\pi) \otimes C^*(G_\pi, \omega_\pi^{-1}) \cong \mathcal{K}(H_\pi) \rtimes_\alpha G_\pi \quad \text{via} \quad k \otimes f \mapsto \left[s \mapsto f(s) k W_s^*\right]$$

and $\quad (\mathcal{K}(H_\pi) \rtimes_\alpha G_\pi)\widehat{\ } = \{(\pi \otimes 1) \rtimes (W \otimes V) : V \in (G_\pi, \omega_\pi^{-1})\widehat{\ }\}.$

Mackey's little group method: An application

Theorem (Mackey-Takesaki-Green) Suppose that A is type I and $\alpha : G \to \mathrm{Aut}(A)$ satisfies the regularity conditions (1)–(3). Then $A \rtimes_\alpha G$ is type I if and only if $C^*(G_\pi, \omega_\pi^{-1})$ is type I for all $\pi \in \widehat{A}$.

Proof $A \rtimes_\alpha G$ is type I if and only if

$$\forall \rho \in (A \rtimes_\alpha G)\widehat{\ } : \quad \rho(A \rtimes_\alpha G) \supseteq \mathcal{K}(H_\rho).$$

Since every ρ belongs to $(A_{G(\pi)} \rtimes_\alpha G)\widehat{\ }$ for one orbit $G(\pi) \subseteq \widehat{A}$, it suffices to show that $A_{G(\pi)} \rtimes_\alpha G$ is type I for all $\pi \in \widehat{A}$. But

$$A_{G(\pi)} \rtimes_\alpha G \sim_M A_\pi \rtimes_\alpha G_\pi \sim_M C^*(G_\pi, \omega_\pi^{-1})$$

and the type I property is preserved by Morita equivalence. □

Note: the theorem always applies when G is compact!

Mackey's little group method: An application

Theorem (Dixmier-Pukanszky) Let G be a real (locally) algebraic group. Then $C^*(G)$ is type I

Rough idea of Proof: Show by induction on $\dim(G)$ that $C^*(G)$ type I. Easy if $\dim(G) = 1$. If $\dim(G) > 1$ there are two cases:
Case 1: G is reductive. Then the result is due to Harish Chandra.

Case 2: Let N denote the nilradical of G. Then $G \cong N \rtimes R$ with N unipotent and R reductive. Thus we may decompose

$$C^*(G) \cong C^*(N) \rtimes_\alpha R.$$

By Kirillov, we know that

$$\widehat{N} \cong \mathfrak{n}^* / \operatorname{Ad}^*(N) \quad \text{hence} \quad \widehat{N}/R = \widehat{N}/G \cong \mathfrak{n}^* / \operatorname{Ad}^*(G).$$

Since the G-action is algebraic, the orbits are locally closed and the Mackey machine applies. Hence

$$C^*(G) \text{ type I} \iff \forall \pi \in \widehat{N}: \; C^*(R_\pi, \omega_\pi^{-1}) \text{ type I}$$

But $\dim(R_\pi) < \dim(G)$ (need some extra care for ω_π^{-1}!)

The Baum-Connes conjecture

The Baum-Connes conjecture predicted that a certain map

$$\mu : K_*^{top}(G, A) \to K_*(A \rtimes_{\mathrm{red}} G)$$

is an isomorphism (not always true, but very often).

For (almost) connected G, this implies the

Connes-Kasparov conjecture

$$K_*(C_r^*(G)) \cong K_*(C^*(V \rtimes K)) \overset{\text{if spin}^c}{\cong} K_{*+\dim(V)}(C^*(K))$$

where $K < G$ maximal compact subgroup, $V = T_{eK}(G/K)$.

A. Wassermann '87, V. Lafforgue '02 The Connes-Kasparov conjecture (i.e., BC for $\mathbb{C}$) holds for all reductive groups G.

Mackey-machine for Baum-Connes

Chabert, E., Oyono-Oyono, Nest 2000–2003
Mackey machine for BC:

(1) Suppose $\alpha : G \curvearrowright A$ such that A is type I and $G \curvearrowright \widehat{A}$ is reegular. Then the following (almost) holds:

$$\forall \pi \in \widehat{A}: \ G_\pi \text{ satisfies BC for } \mathcal{K}(H_\pi) \ \Rightarrow \ G \text{ satisfies BC for } A$$

(2) If $N \lhd G$ is amenable, then

$$G \text{ satisfies } BC \ \Leftrightarrow \ \dot{G} := G/N \text{ satisfies BC for } C^*(N) \otimes \mathcal{K}.$$

Theorem (Chabert-E-Nest '03) The Baum-Connes conjecture (for $C_r^*(G) = \mathbb{C} \rtimes_{\text{red}} G$) holds for all almost connected groups and for all linear algebraic groups over $\mathbb{Q}_p$.

Bibliography

- B. Bekka and S. Echterhoff *On unitary representations of algebraic groups over local fields.* arXiv:2003.02297 (to appear in Representation Theory)
- J. Chabert, S. Echterhoff and R. Nest, *The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups.* Publ. Math. Inst. Hautes Études Sci. **97** (2003), 239–278.
- J. Cuntz, S. Echterhoff, X. Li, and G. Yu. K-theory for group C^*-algebras and semigroup C^*-algebras, Oberwolfach Seminars, 47, Birkhäuser/Springer, Cham, 2017.
- J. Dixmier, *Sur les représentations unitaires des groupes de Lie algébriques*, Ann. Inst. Fourier, Grenoble, **7**, (1957), 315–328.
- P. Green. *The local structure of twisted covariance algebras*. Acta. Math., **140** (1978), 191–250.
- G.W. Mackey, *Unitary representations of group extensions. I*, Acta Math., **99** (1958), 265–311
- Lajos Pukánszky. Characters of connected Lie groups. Mathematical surveys and Monographs Vol 71. American Mathematical Society, Rhode Island 1999.

Thanks for your attention!