Hilbert C^*-modules

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I Definition and examples

Background

Hilbert C^*-modules are generalisations of Hilbert spaces, where the field of scalars $\mathbb{C}$ is replaced by a C^*-algebra A.

They were first defined by Paschke in 1973. They are closely related to K-theory, and key ingredients of the definition of Kasparov's KK-theory.

Pre-Hilbert modules

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ with an anti-linear anti-involution $a \mapsto a^*$. We say that $a \geq 0$ if there is a $b \in \mathcal{A}$ such that $a = b^* b$.

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Definition

A **(right) pre-Hilbert $\mathcal{A}$-module** is a complex vector space $\mathcal{E}$ which is also a (linear) right $\mathcal{A}$-module, together with a map $(-,-)\colon \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that for all $u, v, w \in \mathcal{E}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
(u, v + w) &= (u, v) + (u, w)\\
(v, wa) &= (v, w)a\\
(v, \lambda w) &= \lambda(v, w)\\
(v, w) &= (w, v)^*\\
(v, v) &\geq 0\\
(v, v) = 0 &\Rightarrow v = 0.
\end{aligned}$$

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A pre-Hilbert $\mathbb{C}$-module is a complex inner product space.

Hilbert C^*-modules

From now on, let A be a C^*-algebra, with norm $\| \cdot \|_A$. Let $\mathcal{E}$ a pre-Hilbert A-module.

Definition

The **norm** of $v \in \mathcal{E}$ is

$$\|v\|_{\mathcal{E}} = \sqrt{\|(v, v)\|_A}$$

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Definition

A pre-Hilbert A-module $\mathcal{E}$ is a **(right) Hilbert A-module** if it is complete in the norm $\|\cdot\|_{\mathcal{E}}$.

Example 1: Hilbert spaces

A Hilbert $\mathbb{C}$-module is a Hilbert space.

Example 2: A over itself

View A as a right module over itself by right multiplication.

For $a, b \in A$, define

$$(a, b) = a^* b.$$

Then A becomes a Hilbert A-module.

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Here we use the identity

$$\|a^* a\|_A = \|a\|_A^2.$$

Example 3: A^n

More generally, for any $n \in \mathbb{N}$, the right A-module A^n is a Hilbert A-module, with the inner product

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j^* w_j$$

for $v, w \in A^n$.

Example 4: the standard Hilbert A-module

Let

$$\mathcal{H}_A := \left\{ (a_j)_{j=1}^{\infty}; a_j \in A, \sum_{j=1}^{\infty} a_j^* a_j \text{ converges in } A \right\}.$$

It has a natural right A module structure, and inner product

$$\left((a_j)_{j=1}^{\infty}, (b_j)_{j=1}^{\infty} \right) = \sum_{j=1}^{\infty} a_j^* b_j.$$

This is the **standard Hilbert A-module**.

Example 5: spaces of continuous sections

Let X be a compact Hausdorff space, and $E \to X$ a complex vector bundle. Let $(-,-)_E$ be a Hermitian metric on E, linear in the second entry.

Consider the space $\Gamma(E)$ of continuous sections of E. It is a right $C(X)$-module by pointwise multiplication.

For $s_1, s_2 \in \Gamma(E)$, define $(s_1, s_2) \in C(X)$ by

$$(s_1, s_2)(x) = (s_1(x), s_2(x))_E.$$

Then $\Gamma(E)$ is a Hilbert $C(X)$-module.

Example 6: proper actions

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Let X be a locally compact Hausdorff space, with a proper action by a locally compact group G. Let μ be a G-invariant Borel measure on X, so that every relatively compact open set has finite, nonzero measure.

Let $E \to X$ be a G-vector bundle, with a G-invariant Hermitian metric $(-,-)_E$. For $s_1, s_2, \in \Gamma_c(E)$, define

$$(s_1, s_2)_{L^2(E)} = \int_X (s_1(x), s_2(x))_E \, d\mu(x).$$

Example 6: proper actions (cont'd)

For $g \in G$, $x \in X$ and $s \in \Gamma_c(E)$, define

$$(g \cdot s)(x) = g \cdot (s(g^{-1}x)).$$

- For $f \in C_c(G)$, define

$$s \cdot f = \int_G f(g)(g^{-1} \cdot s)\, dg.$$

- For $s_1, s_2 \in \Gamma_c(E)$, define

$$(s_1, s_2)(g) = (s_1, g \cdot s_2)_{L^2(E)}.$$

  Because the action is proper, this defines

$$(s_1, s_2) \in C_c(G) \subset C^*G.$$

This makes $\Gamma_c(E)$ a pre-Hilbert $C_c(G)$-module. We can complete it to a Hilbert $C^*(G)$-module.

II Operators on Hilbert C^*-modules

Adjointable operators

Let $\mathcal{E}$ be a Hilbert A-module.

Definition

An **adjointable operator** on $\mathcal{E}$ is a map $T\colon \mathcal{E} \to \mathcal{E}$ for which there exists a map $T^*\colon \mathcal{E} \to \mathcal{E}$ such that for all $v, w \in \mathcal{E}$,

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Lemma

The adjointable operators on a $\mathcal{E}$ form a C^*-algebra with respect to the operator norm.

This C^*-algebra is denoted by $\mathcal{L}(\mathcal{E})$.

Example of a non-adjointable bounded module endomorphism

This example is due to Paschke.

Let $A = C([0,1])$ and $J = \{f \in A; f(0) = 0\}$. Consider the Hilbert A-module $\mathcal{E} := J \times A$, with inner product

$$\big((f_1, g_1), (f_1, g_2)\big) = \bar{f}_1 f_2 + \bar{g}_1 g_2,$$

for $f_1, f_2 \in J$ and $g_1, g_2 \in A$.

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for $f_1, f_2 \in J$ and $g_1, g_2 \in A$.

Define $T \colon \mathcal{E} \to \mathcal{E}$ by

$$T(f, g) = (0, f)$$

for $f \in J$, $g \in A$. Then T is a bounded module map, but **not adjointable**.

Example: compact operators?

The compact operators should be the ones that are 'almost finite-rank'.

Example

Let X be a compact Hausdorff space, and consider $C(X)$ as a Hilbert $C(X)$-module. Let $M_f \colon C(X) \to C(X)$ be given by multiplication by a nonzero $f \in C(X)$.

- $C(X)$ is a one-dimensional module over itself, so M_f 'should be' **finite-rank**
- but M_f is **not compact** in the Banach space sense.

Compact operators

Let $\mathcal{E}$ be a Hilbert A-module.

Definition

- The space $\mathcal{F}(\mathcal{E})$ of **finite-rank operators** on $\mathcal{E}$ is spanned by operators of the form $\theta_{v,w}\colon \mathcal{E} \to \mathcal{E}$, for $v, w \in \mathcal{E}$, defined by
$$\theta_{v,w}(u) = v(w, u).$$
- The space $\mathcal{K}(\mathcal{E})$ of **compact operators** on $\mathcal{E}$ is the closure of $\mathcal{F}(\mathcal{E})$ in $\mathcal{L}(\mathcal{E})$.

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Lemma

The space $\mathcal{K}(\mathcal{E})$ is a closed $*$-ideal in $\mathcal{L}(\mathcal{E})$.

Example 1: Hilbert spaces

If $\mathcal{E}$ is a Hilbert $\mathbb{C}$-module (a Hilbert space), then

- the adjointable operators on $\mathcal{E}$ are exactly the bounded ones
- the compact operators on $\mathcal{E}$ are the compact operators in the usual sense.

Example 2: A over itself

For the Hilbert A-module A,

- $\mathcal{L}(A) = \mathcal{M}(A)$, the multiplier algebra of A
- $\mathcal{K}(A) = A$, via left multiplication.

Example 3: A^n

For the Hilbert A-module A^n,

- $\mathcal{L}(A) = M_n(\mathcal{M}(A))$
- $\mathcal{K}(A) = M_n(A)$.

Example 4: $\mathcal{H}_A$

We write $\mathcal{K}$ for the C^*-algebra of compact operators on an infinite-dimensional separable Hilbert space.

For the Hilbert A-module $\mathcal{H}_A$,

- $\mathcal{L}(\mathcal{H}_A) = \mathcal{M}(A \otimes \mathcal{K})$, the multiplier algebra of $A \otimes \mathcal{K}$
- $\mathcal{K}(\mathcal{H}_A) = A \otimes \mathcal{K}$.

Fredholm operators

Definition

An operator $F \in \mathcal{L}(\mathcal{E})$ is A-**Fredholm** if it is invertible modulo $\mathcal{K}(\mathcal{E})$.

Theorem (Mingo, 1987)

Suppose that $1 \in A$. Let F be an A-Fredholm operator on a Hilbert A-module $\mathcal{E}$. Then there is a compact operator $K \in \mathcal{K}(\mathcal{E})$ such that the operator

$$F' := (F + K) \oplus 1 \quad \in \mathcal{L}(\mathcal{E} \oplus \mathcal{H}_A),$$

has closed range, and the A-modules $\ker F'$ and $\ker F'^*$ are finitely generated and projective.

(Kasparov's stabilisation theorem: if $\mathcal{E}$ is separable, then $\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A$.)

Example: proper actions

Consider a locally compact group G acting properly an isometrically on a Riemannian manifold M, and let $E \to M$ be a Hermitian G-vector bundle. Let $\mathcal{E}$ be the Hilbert $C^*(G)$-module defined by completing $\Gamma_c(E)$ in the inner product

$$(s_1, s_2)(g) = (s_1, g \cdot s_2)_{L^2(E)}.$$

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$$(s_1, s_2)(g) = (s_1, g \cdot s_2)_{L^2(E)}.$$

Let D be a first-order, elliptic, self-adjoint differential operator on E. Set

$$\tilde{F} := \frac{D}{\sqrt{D^2 + 1}}.$$

This operator does not preserve $\Gamma_c(E)$.

Example: proper actions (cont'd)

We had

$$\tilde{F} := \frac{D}{\sqrt{D^2+1}}.$$

Now suppose that M/G is compact. Let $\chi \in C_c(M)$ be such that for all $m \in M$,

$$\int_G \chi(gm)^2 \, dg = 1.$$

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Proposition

The operator

$$F = \int_G g\chi\tilde{F}\chi g^{-1} \, dg.$$

preserves $\Gamma_c(E)$, and extends to a $C^*(G)$-Fredholm operator on $\mathcal{E}$.

The operators F and $\tilde{F}$ are homotopic in a suitable sense.

Unbounded operators

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Definition

An unbounded operator T on a Hilbert C^*-module $\mathcal{E}$ is **regular** if

- $\mathrm{dom}(T^*) \subset \mathcal{E}$ is dense
- $\mathrm{im}(1 + T^*T) \subset \mathcal{E}$ is dense.

Example

If $A = \mathbb{C}$, then every closed operator is regular.

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Proposition (Baaj–Julg, 1983)

If T is regular, then $\frac{T}{\sqrt{T^*T+1}}$ is adjointable.

More generally, there is a notion of functional calculus of regular self-adjoint operators.

III Bimodules and tensor products

Bimodules

Let A and B be C^*-algebras.

Definition

A **Hilbert** (A, B)-**bimodule** is a Hilbert B-module $\mathcal{E}$ together with a $*$-homomorphism $A \to \mathcal{L}(\mathcal{E})$.

In particular, a Hilbert (A, B)-bimodule is a left A-module and a right B-module.

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- A Hilbert $(\mathbb{C}, B)$-bimodule is a Hilbert B-module.
- A Hilbert $(A, \mathbb{C})$-bimodule is a $*$-representation of A.

Tensor products

Let

- A, B and C be C^*-algebras
- $\mathcal{E}$ be a Hilbert (A, B)-bimodule, and $\mathcal{F}$ a Hilbert (B, C)-bimodule
- $(-,-)_{\mathcal{E}}$ be the B-valued inner product on $\mathcal{E}$, and $(-,-)_{\mathcal{F}}$ the C-valued inner product on $\mathcal{F}$.

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Consider the algebraic tensor product $\mathcal{E} \otimes \mathcal{F}$ over $\mathbb{C}$. For $v_1, v_2 \in \mathcal{E}$ and $w_1, w_2 \in \mathcal{F}$, define

$$(v_1 \otimes w_1, v_2 \otimes w_2) = \big(w_1, (v_1, v_2)_{\mathcal{E}} w_2\big)_{\mathcal{F}} \quad \in C. \tag{1}$$

In the second entry, we used the left B-module structure on $\mathcal{F}$.

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Definition

The **tensor product of $\mathcal{E}$ and $\mathcal{F}$ over B** is the completion of $\mathcal{E} \otimes \mathcal{F}$ modulo the vectors of norm 0 in the inner product (1). It is denoted by

$$\mathcal{E} \otimes_B \mathcal{F}.$$

This tensor product is a Hilbert (A, C)-bimodule.

Tensor products (cont'd)

For all $v \in \mathcal{E}$, $w \in \mathcal{F}$ and $b \in B$,

$$\langle vb \otimes w - v \otimes bw, vb \otimes w - v \otimes bw \rangle = 0.$$

Hence elements of the form $vb \otimes w - v \otimes bw$ are divided out in the definition of $\mathcal{E} \otimes_B \mathcal{F}$.

Example 1: pullbacks of vector bundles

Let $f\colon X \to Y$ be a continuous map between compact Hausdorff spaces. Let $E \to Y$ be a complex vector bundle. Then

- $\Gamma(E)$ is a Hilbert $C(Y)$-module, i.e. a Hilbert $(\mathbb{C}, C(Y))$-bimodule
- $C(X)$ is a Hilbert $C(X)$-module, and the pullback

  $$f^*\colon C(Y) \to C(X) \to \mathcal{L}(C(X))$$

  makes $C(X)$ a Hilbert $(C(Y), C(X))$-bimodule.

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  makes $C(X)$ a Hilbert $(C(Y), C(X))$-bimodule.

Now

$$\Gamma(E) \otimes_{C(Y)} C(X) = \Gamma(f^*E),$$

a Hilbert $C(X)$-module.

Example 2: Rieffel induction

This example is due to Rieffel.

Let G be a locally compact group, and $H < G$ a closed subgroup. Fix left Haar measures, and let δ_G and δ_H be the modular functions of G and H, respectively.

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Let G be a locally compact group, and $H < G$ a closed subgroup. Fix left Haar measures, and let δ_G and δ_H be the modular functions of G and H, respectively.

Consider the right action by $C_c(H)$ on $C_c(G)$ given by

$$(f_G f_H)(g) = \int_H \frac{\delta_G(h)^{1/2}}{\delta_H(h)^{1/2}} f_G(gh^{-1}) f_H(h)\, dh.$$

Consider the $C_c(H)$-valued inner product on $C_c(G)$ given by

$$(f_1, f_2)(h) = \frac{\delta_G(h)^{1/2}}{\delta_H(h)^{1/2}} \int_G \bar{f}_1(g^{-1}) f_2(g^{-1}h)\, dg.$$

This makes $C_c(G)$ a pre-Hilbert $C_c(H)$-module. Let $\mathcal{E}_H^G$ be its completion to a Hilbert $C^*(H)$-module.

Example 2: Rieffel induction (cont'd)

The Hilbert $C^*(H)$-module $\mathcal{E}_H^G$ has a left action by $C^*(G)$ that extends the left convolution action by $C_c(G)$ on itself. This makes $\mathcal{E}_H^G$ a Hilbert $(C^*(G), C^*(H))$-bimodule.

Example 2: Rieffel induction (cont'd)

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In this way, every $*$-representation V of $C^*(H)$ gives an induced $*$-representation

$$\mathcal{E}_H^G \otimes_{C^*(H)} V$$

of $C^*(G)$.

Theorem (Rieffel, 1974)

This implements Mackey induction.

Parabolic induction: a modular function

The following construction is due to Pierre Clare.

Let G be a connected, linear, real semisimple Lie group, and $P < G$ a cuspidal parabolic subgroup, with Langlands decomposition $P = MAN$. Set $L = MA$.

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Fix left Haar measures dg on G, and dn on N. Let $d(gN)$ be the G-invariant measure on G/N such that for all $f \in C_c(G)$,

$$\int_{G/N} \int_N f(gn)\, dn\, d(gN) = \int_G f(g)\, dg.$$

Let $\delta\colon L \to \mathbb{R}_+$ be such that for all $f \in C_c(G/N)$ and $l \in L$,

$$\int_{G/N} f(glN)\, d(gN) = \delta(l)^{-1} \int_{G/N} f(gN)\, d(gN).$$

Then $\delta(l) = |\det(\mathrm{Ad}(l)\colon \mathfrak{n} \to \mathfrak{n})| = e^{2\rho(\log a)}$ if $l = ma \in L$.

Parabolic induction: a Hilbert $C^*(L)$-module

Consider the right action by $C_c^\infty(L)$ on $C_c^\infty(G/N)$ given by

$$(f_{G/N} f_L)(gN) = \int_L \delta(l)^{1/2} f_{G/N}(glN) f_L(l^{-1})\, dl.$$

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For $f_1, f_2 \in C_c^\infty(G/N)$, define $(f_1, f_2) \in C_c^\infty(L)$ by

$$(f_1, f_2)(l) = \delta(l)^{1/2} \int_{G/N} \bar{f}_1(gN) f_2(glN)\, d(gN).$$

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Definition

The Hilbert $C^*(L)$-module $C^*(G/N)$ is the completion of the pre-Hilbert $C_c^\infty(L)$-module $C_c^\infty(G/N)$.

Parabolic induction: action by $C^*(G)$

For $f_G \in C_c^\infty(G)$, $f_{G/N} \in C_c^\infty(G/N)$ and $g \in G$, define

$$(f_G f_{G/N})(gN) = \int_G f_G(g') f_{G/N}(g'^{-1} gN)\, dg'.$$

This extends to a $*$-homomorphism $C^*(G) \to \mathcal{L}(C^*(G/N))$.

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This extends to a $*$-homomorphism $C^*(G) \to \mathcal{L}(C^*(G/N))$.

In this way, $C^*(G/N)$ becomes a Hilbert $(C^*(G), C^*(L))$-bimodule.

Parabolic induction as a tensor product

Consider a unitary irreducible representation π of L on a Hilbert space H. Then π defines a $*$-representation

$$\pi\colon C^*(L) \to \mathcal{L}(H)$$

by continuous extension of

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Theorem (Clare, 2013)

The tensor product

$$C^*(G/N) \otimes_{C^*(L)} H$$

is the induced representation $\mathrm{Ind}_P^G(\pi)$*, viewed as a* $*$*-representation of* $C^*(G)$*.*

Morita equivalence

Let A and B be C^*-algebras.

Definition

The algebra A is **strongly Morita equivalent** to B if there is a Hilbert (A, B)-bimodule $\mathcal{E}$ such that

- $\mathrm{span}\{(v, w); v, w \in \mathcal{E}\}$ is dense in B
- $\mathcal{K}(\mathcal{E}) \cong A$.

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Proposition (Rieffel, 1974)

Morita equivalence is an equivalence relation on C^*-algebras.

Stable isomorphism

Example

We view $\mathcal{H}_A$ as an $(A \otimes \mathcal{K}, A)$-bimodule. Then

- $\text{span}\{(v, w); v, w \in \mathcal{H}_A\}$ contains $\text{span}\{a^*b; a, b \in A\}$ and is dense in A
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So $A \otimes \mathcal{K}$ is strongly Morita equivalent to A.

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Theorem (Brown, 1977)

- *If $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, then A and B are strongly Morita equivalent.*
- *If A and B are strongly Morita equivalent and have countable approximate units, then $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.*

Morita equivalence and representations

Definition

A **Hermitian** A**-module** is a $*$-representation $\pi \colon A \to \mathcal{B}(H)$ in a Hilbert space H, such that

$$\{\pi(a)v; a \in A, v \in H\} \subset H$$

is dense. The category of such modules and bounded module homomorphisms is denoted by $\mathrm{HMod}(A)$.

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Theorem (Rieffel)

If A and B are strongly Morita equivalent, then tensoring with the corresponding bimodules defines an equivalence of categories between $\mathrm{HMod}(A)$ *and* $\mathrm{HMod}(B)$*.*

More information

General background:

- E. Christopher Lance, *Hilbert C^*-modules: a toolkit for operator algebraists*, Cambridge University Press, 1995.

Relations to $K(K)$-theory:

- Bruce Blackadar, *K-theory for operator algebras*, second edition, Cambridge University Press, 1998.
- Nigel Higson, *A primer on KK-theory*, Proc. Sympos. Pure Math., 51, Part 1, AMS, 1990.
- N.E. Wegge–Olsen, *K-theory and C^*-algebras*, Oxford Science Publications, 1993.

Thank you