A quick introduction to C*-algebras

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C*-algebras

A C^*-algebras is a Banachalgebra with involution $a \mapsto a^*$ such that

$$\|a^*a\| = \|a\|^2 \quad \forall a \in A.$$

Examples

- Let X a locally compact Hausdorff space

$$C_0(X) = \{f : X \to \mathbb{C} : f \text{ continuous}, \ f(\infty) = 0\}$$

$$\|f\|_\infty = \sup_{x \in X} \|f(x)\| \quad \text{and} \quad f^* := \bar{f}$$

- Let H be a Hilbert space

$$\mathcal{B}(H) := \{T : H \to H : \|T\|_{op} < \infty\} \quad \langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$$

$$\|T\|_{op} := \sup_{\|\xi\| \leq 1} \|T\xi\|.$$

The first Gelfand-Naimark theorem

Theorem (Gelfand-Naimark 1943)

Every commutative C^*-algebra is (isometrically) $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

Idea of Proof Define

$$X = \widehat{A} := \{\chi : A \to \mathbb{C} : 0 \neq \chi \text{ an algebra homom.}\}.$$

$\widehat{A}$ is locally closed (=open in its closure) in $B_1(A')$, hence locally compact by Banach-Alaoglu (compact iff A is unital).
Then $A \cong C_0(\widehat{A})$ via

$$\widehat{\ } : A \to C_0(\widehat{A}); a \mapsto \widehat{a}, \text{ with } \widehat{a}(\chi) := \chi(a).$$

□

Philosophy: View noncommutative C*-algebras as 'noncommutative topological spaces'.

2nd Gelfand-Naimark theorem

Theorem (Gelfand-Naimark 1943)

Every C^*-algebra is (isometrically) $*$-isomorphic to a closed $*$-subalgebra of $\mathcal{B}(H)$ for some H.

Idea of Proof Consider the *state space* of A:

$$S(A) := \{\varphi : A \to \mathbb{C} : \varphi \text{ linear}; \varphi(a^*a) \geq 0 \ \forall a \in A, \|\varphi\| = 1\}.$$

Then $\forall a \in A \ \exists \varphi \in S(A)$ such that $\varphi(a^*a) > 0$. Define $\forall a, b \in A$:

$$\langle a, b\rangle_\varphi := \varphi(a^*b), \quad N_\varphi = \{a : \langle a, a\rangle_\varphi = 0\}, \quad H_\varphi := \overline{A/N_\varphi}^{\langle\cdot,\cdot\rangle_\varphi}$$

and

$$\pi_\varphi : A \to \mathcal{B}(H_\varphi); \quad \pi_\varphi(a)(b + N_\varphi) := ab + N_\varphi$$

Then we get a faithful representation

$$\pi := \bigoplus_{\varphi \in S(A)} \pi_\varphi : A \to \mathcal{B}(\bigoplus_\varphi H_\varphi)$$

Unitizations: The Stone Čech compactification

Problem: A may not be unital (e.g. $A = C_0(X)$, X not. cpct.).

Define $M(A)$ to be the set of all maps $m : A \to A$ such that

$$\exists m^* : A \to A \quad \text{satisfying} \quad \forall a, b \in A : am(b) = (m^*(a^*))^* b.$$

One checks that $m(a)b = m(ab)$ $\forall a, b \in A$ and m, m^* are bounded. $M(A)$ becomes a unital C^*-algebra with respect to

$$mn := m \circ n, \quad m \mapsto m^*, \quad \text{and} \quad \|m\| := \|m\|_{op}.$$

Moreover: $A \lhd M(A)$ via $a \mapsto (b \mapsto ab)$ and then $ma = m(a)$.

$M(A)$ is the largest unital C^*-algebra which contains A as an essential ideal, that is $m = 0 \Leftrightarrow ma = 0 \quad \forall a \in A$.

Universal property
If $A \lhd M$ then $\exists\, \iota : M \to M(A); \iota(m)(a) := ma$,
which is faithful if $A \lhd M$ essential

Definition We call $M(A)$ the multiplier algebra of A.

Unitizations: The Stone Čech compactification

Examples

1. We have $M(C_0(X)) \cong C_b(X) \cong C(\beta(X))$.
2. If A is unital, then $M(A) = A$ (since $m = m(1) \in A$).
3. If H is a Hilbert space, then $M(\mathcal{K}(H)) = \mathcal{B}(H)$.
4. If $\pi : A \hookrightarrow \mathcal{B}(H)$ is a nondegenerate (i.e. $\pi(A)H = H$) and faithful $*$-representation, then

$$M(A) \cong \{T \in \mathcal{B}(H) : T\pi(A) \cup \pi(A)T \subseteq \pi(A)\}.$$

Some facts: **(a)** $M(A)$ is the strict completion of A with strict topology generated by the seminorms $m \mapsto \|m(a)\|, \|m^*(a)\|$.

(b) If $\Phi : A \to M(B)$ is a nondeg. $*$-hom. (i.e., $\Phi(A)B = B$),

$$\exists! \;\; \bar{\Phi} : M(A) \to M(B) \quad \text{s.t.} \quad \bar{\Phi}(m)(\Phi(a)b) := \Phi(ma)b.$$

$\bar{\Phi}$ is strictly continuous, and faithful iff Φ is faithful.

The one point compactification and functional calculus

Define $A^+ := A + \mathbb{C}1 \subseteq M(A)$ (the smallest unitization of A)

Then $C_0(X)^+ = C(X^+)$ with $X^+ := X \cup \{\infty\}$.

For $a \in A$ define $\sigma(a) := \{\lambda \in \mathbb{C} : a + \lambda 1 \text{ not invertible in } A^+\}$.

If $a \in A$ is normal, i.e. $a^*a = aa^*$ then $C^*(a, 1) \subseteq A^+$ is commutative, and

$$\widehat{C^*(a,1)} \cong \sigma(a) \quad \text{via} \quad \chi \mapsto \chi(a)$$

The inverse of the Gelfand map $C^*(a, 1) \cong C(\widehat{C^*(a,1)})$ gives an isometric $*$-homomorphism

$$\Phi : C(\sigma(a)) = C(\widehat{C^*(a,1)}) \to C^*(a, 1) : f \mapsto f(a)$$

One checks that $f(a) \in C^*(a) \subseteq A$ if $f(0) = 0$!

Important: This is the main tool for many constructions in C^*-theory (square roots, approximate units, positivity, etc..)

The maximal tensor product

Let A, B be C^*-algebras, $A \odot B$ the algebraic tensor product with multiplication and involution

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \quad \text{and} \quad (a \otimes b)^* = a^* \otimes b^*.$$

If $\pi : A \to M(C), \rho : B \to M(C)$ s.t. $\pi(a)\rho(b) = \rho(b)\pi(a)\ \forall a, b$. Then there exists a $*$-homom.

$$\pi \times \rho : A \odot B \to M(C); \sum a_i \otimes b_i \mapsto \sum \pi(a_i)\rho(b_i)$$

For $x \in A \odot B$ define

$$\|x\|_{\max} := \sup_{\pi \times \rho} \|\pi \times \rho(x)\| \quad \text{and} \quad A \otimes_{\max} B := \overline{A \odot B}^{\|\cdot\|_{\max}}.$$

By construction, every $\pi \times \rho$ extends uniquely to $A \otimes_{\max} B$!

Conversely: let $i_A, i_B : (A, B) \hookrightarrow M(A \otimes_{\max} B)$ given by

$$i_A(a)(a' \otimes b') = aa' \otimes b \quad \text{and} \quad i_B(b)(a' \otimes b') = a \otimes bb'.$$

If $\Phi : A \otimes_{\max} B \to M(C)$ any nondeg. $*$-homomorphism, then

$$\Phi = \pi \times \rho \quad \text{with} \quad \pi := \bar{\Phi} \circ i_A,\ \rho := \bar{\Phi} \circ i_B.$$

The minimal tensor product

Let $\pi : A \to \mathcal{B}(H), \rho : B \to \mathcal{B}(K)$ be faithful $*$-representations.

$\pi \otimes \rho : A \odot B \to \mathcal{B}(H \hat{\otimes} K)$; $\pi \otimes \rho(a \otimes b)(\xi \otimes \eta) = \pi(a)\xi \otimes \rho(b)\eta$.

Takesaki '64 The Norm $\|x\|_{\min} := \|\pi \otimes \rho(x)\|$ does not depend on the choices π and ρ and it is the smallest C^*-cross norm on $A \odot B$.

$A \otimes_{\min} B = \overline{A \odot B}^{\|\cdot\|_{\min}}$ is called the minimal tensor product.

Definition

A C^*-algeba A is called nuclear if $\forall B: \; A \otimes_{\max} B = A \otimes_{\min} B$.

Examples

(a) $C_0(X)$ is nuclear with $C_0(X) \otimes B = C_0(X, B)$ and $C_0(X) \otimes C_0(Y) = C_0(X \times Y)$.

(b) $\mathcal{K}(H)$, type I C^*-algebras, $C^*(G)$ if G connected or amenable are nuclear.

(c) The following are not nuclear: $\mathcal{B}(H)$, $C^*(\mathbb{F}_2)$, $C^*(\Gamma)$ for Γ discrete non-amenable!

(twisted) Group C*-algebras

Let G be a locally compact group, and $\omega : G \times G \to \mathbb{T}$ a Borel 2-cocycle, i.e. ω is a Borel map such that $\forall s, t \in G$:

$$\omega(s,t)\omega(st,r) = \omega(s,tr)\omega(t,r) \quad \text{and} \quad \omega(s,e) = 1 = \omega(e,s)$$

Let $L^1(G,\omega) := L^1(G)$ equipped with

$$f *_\omega g(s) = \int_G f(t)g(s^{-1}t)\omega(t,t^{-1}s)\,dt,\ \ f^*(s) = \Delta(s^{-1})\overline{\omega(s,s^{-1})f(s^{-1})}.$$

An ω-representation is a strictly Borel map $V : G \to UM(B)$, s.t.

$$V_s V_t = \omega(s,t) V_{st}.$$

It integrates to $\tilde{V} : L^1(G,\omega) \to M(B)$; $\tilde{V}(f) = \int_G f(t)V_t\,dt$.
Define $C^*(G,\omega) := \overline{L^1(G,\omega)}^{\|\cdot\|_{\max}}, \quad \|f\|_{\max} = \sup_V \|\tilde{V}(f)\|$.

(twisted) Group C^* algebras

Observation: there is canonical ω-representation

$$i_G : G \to UM(C^*(G,\omega)), \quad (i_G(s)(f))(t) = \omega(s, s^{-1}t)f(s^{-1}t).$$

Then: If $\Phi : C^*(G,\omega) \to M(B)$ is a nondeg. $*$-representation we get

$$\Phi = \tilde{V} \quad \text{for} \quad V = \Phi \circ i_G.$$

We therefore get a one-to-one correspondence between nondeg $*$-reps of $C^*(G,\omega)$ and ω-unitary reps of G!

Note: If $B = \mathcal{K}(H)$, then $M(B) = \mathcal{B}(H)$ and $UM(B) = \mathcal{U}(H)$ Hence this correspondence covers representations on Hilbert spaces and it preserves irreducibilty and unitary equivalence in both directions!

If $\omega \equiv 1$ we get the maximal (or full) group C^*-algebra $C^*(G)$, which is universal for unitary representations $U : G \to UM(B)$ (strictly cont. homomorphisms). We also get a one-to-one correspondence

$$\widehat{G} \longleftrightarrow \widehat{C^*(G)}.$$

Examples

1. Each $\omega \in Z^2(\mathbb{Z}, \mathbb{T})$ is equivalent to one of

$$\omega_\Theta(n, m) = e^{2\pi i \langle \Theta n, m\rangle}, \quad \Theta \in M_n(\mathbb{R}) \text{ s.t. } \Theta^t = -\Theta.$$

   Then $C^*(\mathbb{Z}^n, \omega_\Theta)$ is an *n-dimensional non-commutative torus*

$$C^*(\mathbb{Z}^n, \omega_\Theta) = C^*(u_1, \dots, u_n : u_i \text{ unitary } u_i u_j = e^{2\pi i \Theta_{ij}} u_j u_i)$$

   Note: If $\omega \equiv 1$ we get $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$.

2. $G = \mathbb{R}^n$ we have $\omega \sim \omega_\Theta$ as above and

$$C^*(\mathbb{R}^n, \omega_\Theta) \cong C_0(\mathbb{R}^k) \otimes \mathcal{K}(L^2(\mathbb{R}^m))$$

   for some k, m with $\mathbb{R}^k$ the radical of Θ, $k + 2m = n$.

   If Θ is totally skew (i.e. $k = 0$): $C^*(\mathbb{R}^n, \omega_\Theta) = \mathcal{K}(L^2(\mathbb{R}^m))$

reduced (twisted) Group C^*-algebras

Definition Let (G, ω) be given. Then

$$\lambda_\omega : G \to \mathcal{U}(L^2(G)); \quad (\lambda_\omega(s)\xi)(t) := \omega(s, s^{-1}t)\xi(s^{-1}t).$$

is called the left ω-regular representation of G. We call

$$C_r^*(G, \omega) := \lambda_\omega(C^*(G, \omega)) \subseteq \mathcal{B}(L^2(G))$$

the reduced ω-twisted group C^*-algebra of G.

Note: If G amenable, then $\lambda_\omega : C^*(G, \omega) \overset{\cong}{\to} C_r^*(G, \omega)$.

If $\omega \equiv 1$ we have $C^*(G) = C_r^*(G) \iff G$ amenable.

C*-dynamical systems

Let Aut(A) denote the group of $*$-automorphism of A. An action

$$\alpha : G \to \mathrm{Aut}(A); s \mapsto \alpha_s$$

is a group homom such that $s \mapsto \alpha_s(a)$ is continuous $\forall a \in A$.

Examples

(1) If $G \curvearrowright X; (s, x) \mapsto sx$ is an action of G on X, then

$$\tau : G \to \mathrm{Aut}(C_0(X)); (\tau_s(f))(x) := f(s^{-1}x).$$

is a corresponding action on $C_0(X)$.

(2) Let N be a closed normal subgroup of G. Then there exists a decomposition action

$$\alpha : G \to \mathrm{Aut}(C^*(N)); \quad (\alpha_s(f))(n) = \delta(s)f(s^{-1}ns), \quad f \in L^1(N)$$

with $\delta(s) = \Delta_G(s)\Delta_{G/N}(s^{-1})$.

Crossed products

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action. Let

$$C_c(G, A) := \{f : G \to A : f \text{ cont. with } \mathrm{supp}(f) \text{ compact}\}$$

equipped with convolution and involution

$$f * g(s) = \int_G f(s)\alpha_s(g(s^{-1}t))\,ds \quad f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1}))^*.$$

A covariant representation $(\pi, U) : (A, G) \to M(B)$ consists of a $*$-rep $\pi : A \to M(B)$ and a unitary rep. $U : G \to UM(B)$ such that

$$\forall a \in A, s \in G : \quad \pi(\alpha_s(a)) = U_s\pi(a)U_s^*.$$

Then (π, U) integrates to

$$\pi \times U : C_c(G, A) \to M(B); \quad \pi \times U(f) = \int_G \pi(f(s)U_s\,ds$$

Define

$$A \rtimes_\alpha G := \overline{C_c(G, A)}^{\|\cdot\|_{\max}} \quad \text{with} \quad \|f\|_{\max} := \sup_{(\pi,U)} \|\pi \times U(f)\|.$$

Crossed products

Recall: The full (or maximal) crossed product $A \rtimes_\alpha G$ is defined as

$$A \rtimes_\alpha G := \overline{C_c(G, A)}^{\|\cdot\|_{\max}} \quad \text{with} \quad \|f\|_{\max} := \sup_{(\pi, U)} \|\pi \times U(f)\|.$$

Then by construction, every covariant rep (π, U) integrates to

$$\pi \rtimes U : A \rtimes_\alpha G \to M(B).$$

Conversely, $\exists (i_A, i_G) : (A, G) \to M(A \rtimes_\alpha G)$ given by

$$(i_A(a)f)(s) := af(s) \quad (i_G(s)f)(t) := \alpha_s(f(s^{-1}t)), \quad f \in C_c(G, A)$$

If $\Phi : A \rtimes_\alpha G \to M(B)$ is any nondeg. $*$-homomorphism, then

$$\Phi = \pi \rtimes U \quad \text{with} \quad \pi = \bar{\Phi} \circ i_A, \; U = \bar{\Phi} \circ i_G.$$

Reduced crossed product

The regular representation is the integrated form

$$\Lambda := \Lambda_A \rtimes \Lambda_G : A \rtimes_\alpha G \to M(A \otimes \mathcal{K}(L^2(G)))$$

with (Λ_A, Λ_G) defined by $\Lambda_G = 1_A \otimes \lambda_G$ and

$$\Lambda_A : A \xrightarrow{\tilde{\alpha}} M(A \otimes C_0(G)) \xrightarrow{\mathrm{id}_A \otimes M} M(A \otimes \mathcal{K}(L^2(G)))$$

with $\tilde{\alpha}(a) \in C_b(G, A) \subseteq M(A \otimes C_0(G)); \quad \tilde{\alpha}(a)(s) = \alpha_{s^{-1}}(a))$

We define the reduced crossed product

$$A \rtimes_{\mathrm{red}} G := \Lambda(A \rtimes_\alpha G) \subseteq M(A \otimes \mathcal{K}(L^2(G))).$$

If $\pi : A \to \mathcal{B}(H)$ is faithful, then we get a faithful representation

$$\mathrm{Ind}\,\pi : A \rtimes_{\mathrm{red}} G \subseteq M(A \otimes \mathcal{K}(L^2(G))) \xrightarrow{\pi \otimes \mathrm{id}_{\mathcal{K}}} \mathcal{B}(H \hat{\otimes} L^2(G)).$$

Alternatively: $\mathrm{Ind}\,\pi = \tilde{\pi} \rtimes (1 \otimes \lambda_G)$ with

$$(\tilde{\pi}(a)\xi)(s) := \pi(\alpha_{s^{-1}}(a))\xi(s) \quad \xi \in L^2(G, H) \cong H \hat{\otimes} L^2(G).$$

Crossed products

Some facts

1. If B is any C^*-algebra, then

$$(A \rtimes_\alpha G) \otimes_{\max} B \cong (A \otimes_{\max} B) \rtimes_{\alpha \otimes \mathrm{id}_B} G$$

and $$(A \rtimes_{\mathrm{red}} G) \otimes_{\min} B \cong (A \otimes_{\min} B) \rtimes_{\alpha \otimes \mathrm{id},\mathrm{red}} G$$

2. If G amenable, then $A \rtimes_\alpha G = A \rtimes_{\mathrm{red}} G$.
3. G amenable and A nuclear $\Longrightarrow A \rtimes_\alpha G$ is nuclear, since

$$\begin{aligned}
&(A \rtimes_\alpha G) \otimes_{\max} B \cong (A \otimes_{\max} B) \rtimes_{\alpha \otimes \mathrm{id}} G \\
&\cong (A \otimes_{\min} B) \rtimes_{\alpha \otimes \mathrm{id}} G \cong (A \otimes_{\min} B) \rtimes_{\mathrm{red}} G \\
&\cong (A \rtimes_{\mathrm{red}} G) \otimes_{\min} B \cong (A \rtimes_\alpha G) \otimes_{\min} B.
\end{aligned}$$

Other constructions

One can attach C^*-algebras to all kind of mathematical objects, such as

1. groupoids and groupoid actions.
2. partial actions.
3. semigroups and rings
4. graphs and higher rank graphs.
5. coarse metric spaces.
6. ....

and the structure of the algebras reflects the structure of the mathematical objects.

Dual spaces and the Fell topology

Let $\mathrm{Rep}(A) := \{\pi : A \to \mathcal{B}(H_\pi) : \ast\text{-rep}\}/\sim$ and

$$\widehat{A} := \{\pi \in \mathrm{Rep}(A) : \pi \text{ irreducible}\}.$$

If G is a loc cpct group: $\mathrm{Rep}(G) \leftrightarrow \mathrm{Rep}(C^*(G))$ and $\widehat{G} \leftrightarrow \widehat{C^*(G)}$

Definition If $\pi \in \mathrm{Rep}(A)$ and $E \subseteq \mathrm{Rep}(A)$ we define

$$\pi \prec E \iff \ker \pi \supseteq \bigcap_{\rho \in E} \ker \rho.$$

We then say π is weakly contained in E. Restricted to $\widehat{A}$ we get

$$\pi \in \bar{E} \Leftrightarrow \pi \prec E.$$

Similarly, in $\mathrm{Prim}(A) := \{\ker \pi : \pi \in \widehat{A}\}$ we have the closure operation

$$P \in \overline{E} \Leftrightarrow P \supseteq \bigcap_{Q \in E} Q.$$

These topologies often have very poor separation properties, very often $\widehat{A}$ is not even T_0! But $\mathrm{Prim}(A)$ is always T_0!

Fell-topology

For $\pi \in \mathrm{Rep}(A)$ nongeg. (resp. $\pi \in \mathrm{Rep}(G)$) and $\xi \in H_\pi$ with $\|\xi\| = 1$ let

$$\varphi_{\pi,\xi}(a) = \langle \xi, \pi(a)\xi \rangle \quad \text{(resp. } \varphi_{\pi,\xi}(g) = \langle \xi, \pi(g)\xi \rangle\text{)}.$$

be a state (resp. positive definite funct) associated to π. Then

Theorem (Fell 1960's) The following are equivalent:

1. $\pi \prec E$
2. Every state associated to π is a weak*-limit of states associated to E.

If π is irreducible, these are equivalent to

3. $\exists$ a state associated to π which is a weak*-limit of states associated to E.

If $A = C^*(G)$, then states can be replaced by positive definite functions and weak*-convergence by uniform convergence on compact subsets of G.

Fell-topology

Suppose $I \lhd A$ is a closed ideal. Then if $\pi \in \widehat{A}$ we get either

$$\pi|_I \in \widehat{I} \quad \text{or} \quad \pi(I) = \{0\}$$

In the latter case $\pi \in \widehat{A/I}$ and $\quad \widehat{A} = \widehat{I} \dot{\cup} \widehat{A/I}$.

Similarly $\quad \mathrm{Prim}(A) = \mathrm{Prim}(I) \dot{\cup} \mathrm{Prim}(A/I)$.

The sets $\widehat{I}$ (resp. $\widehat{A/I}$) are the open (resp. closed) subsets of $\widehat{A}$ and similar relations hold for $\mathrm{Prim}(A)$!

Recall that a set E is called locally closed if E is open in its closure!

Important fact: $E \subset \mathrm{Prim}(A)$ is locally closed if and only if there exist closed ideals $I \subseteq J \subseteq A$ such that $E = \mathrm{Prim}(J/I)$.

Simple C^*-algebras

Definition A C^*-algebra is called simple if $\{0\}$ and A are the only closed ideals in A.

If all points $P \in \mathrm{Prim}(A)$ are locally closed, then each point determines a simple subquotient J/I of A s.t. $\{P\} = \mathrm{Prim}(J/I)$.

The simple C^*-algebras can be viewed as the building blocks of general C^*-algebras! (Elliott classification programme!)

Examples

1. $M_n(\mathbb{C})$ and $\mathcal{K}(H)$ are simple.
2. Let $\Theta \in M_n(\mathbb{R})$ with $\Theta^t = -\Theta$. Then $C^*(\mathbb{Z}^n, \omega_\Theta)$ is simple if Θ is totally skew:
$$\forall k \in \mathbb{Z}^n : \big(\forall m \in \mathbb{Z}^n : \langle k, \Theta m\rangle = 0\big) \Rightarrow k = 0.$$
3. If $G \curvearrowright X$ free and minimal, then $C_0(X) \rtimes_{\mathrm{red}} G$ is simple.

Theorem (Lüdeking-Poguntke '94) The simple subquotients of $C^*(G)$ for a connected G are $\mathcal{K}(H)$ or $\mathcal{K}(H) \otimes C^*(\mathbb{Z}^n, \omega_\Theta)$.

Type I C^*-algebras

Definition A C^*-algebra A is called type I (or GCR, or postliminal) if

$$\forall\, \pi \in \widehat{A}: \quad \pi(A) \subseteq \mathcal{K}(H_\pi) \neq \emptyset.$$

And A is called CCR (or liminal) if $\quad \forall\, \pi \in \widehat{A}: \quad \pi(A) = \mathcal{K}(H_\pi).$

Theorem (Glimm) The following are equivalent (for A separable)

1. A is type I.
2. the map $\ker : \widehat{A} \to \mathrm{Prim}(A);\ \pi \mapsto \ker \pi$ is a bijection.
3. $\widehat{A}$ is a T_0-space.
4. $\widehat{A}$ is almost Hausdorff (every nonempty closed set contains a dense open Hausdorff subset).

Examples The following groups have type I group algebras:

1. Motion groups, connected nilpotent groups.
2. reductive groups over local fields (Harish Chandra, Bernstein)
3. real (locally) algebraic groups (Dixmier, Pukanszky).
4. algebraic groups over $\mathbb{Q}_p$ (Bekka-E '21)

Continuous-trace C^*-algebras

Definition A is called a continuous-trace C^*-algebra, if $\widehat{A}$ is Hausdorff, and

$\forall \pi \in \widehat{A}\, \exists \pi \in U \subseteq \widehat{A}$ and $p \in A$ such that $\forall \rho \in U : \rho(p)$ is a rank-one projection.

Dixmier-Douady Let A be a separable and continuous trace with $X := \widehat{A}$. Then there exists a locally trivial bundle $p : \mathcal{X} \to X$ with fibres $p^{-1}(x) = \mathcal{K}(H_x)$ for all $x \in X$ such that

$$A \otimes \mathcal{K}(\ell^2) \cong \Gamma_0(X, \mathcal{X})$$

Then: the stable continuous trace algebras with fixed spectrum $X = \widehat{A}$ are classified by

$$\check{H}^1(X, \mathcal{P}U) \cong \check{H}^2(X, \mathbb{T}) \cong \check{H}^3(X, \mathbb{Z}).$$

Theorem (Dixmier) For every type I C^*-algebra A there is an ascending series of closed ideals (I_ν) over the ordinal numbers ν s.t. $I_{\nu+1}/I_\nu$ is continuous trace for all ν and $A = I_{\nu_0}$ for some ν_0.

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Thanks for your attention!