Conversations on mind, matter, and mathematics

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Remarks

on

Jean-Pierre Changeux & Alain Connes

Conversations on Mind, Matter, and Mathematics [1]

by

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INTRODUCTION

What exactly is the type of reality of mathematical idealities? This problem remains largely an open question. Any ontology of abstract entities will encounter certain aporia which have been well known for centuries if not millenia. These aporia have led the various schools of contemporary epistemology to increasingly deny any reality to mathematical idealities (objects, structures, constructions, proofs) and to justify this denial philosophically, thus rejecting the spontaneous Platonism of most professional mathematicians (however brilliant they may be). With a few rare exceptions, the dominant epistemology of mathematics gives hardly any credence to the thinking of such figures as Poincaré, Husserl, Weyl, Borel, Lebesgue, Veronese, Enriques, Cavaillès, Lautman, Gonseth, or the last Gödel. It is no longer an epistemology of mathematical contents. For quite serious and precise philosophical reasons, it refuses to take into account what the great majority of creative mathematicians consider to be the true nature of mathematical knowledge. And yet, to quote the subtitle of Hao Wang's (1985) book *Beyond Analytic Philosophy*, one might well ask whether the imperative of any valid epistemology should not be: "doing justice to what we know."

The remarkable debate *Conversations on Mind, Matter, and Mathematics* between Alain Connes and Jean-Pierre Changeux, both scientific minds of the very first rank and professors at the Collège de France in Paris, takes up the old question of the reality of mathematical idealities in a rather new and refreshing perspective. To be sure, since it is designed to be accessible to a wide audience, the debate is not framed in technical terms; the arguments often employ a broad brush and are not always

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sufficiently developed. Nevertheless, thanks to the exceptional standing of the protagonists, the debate manages to be compelling and relevant.

I. JEAN-PIERRE CHANGEUX'S NEURAL MATERIALISM

Let us begin by summarizing some of Jean-Pierre Changeux's arguments.

Since mathematics is a human and cognitive activity, it is natural to first analyze it in psychological and neuro-cognitive terms. Psychologism, which formalists and logicists have decried since the time of Frege and Husserl, develops the reductionist thesis that mathematical objects and the logical idealities that formulate them can be reduced—as far as their reality is concerned—to mental states and processes. Depending on whether or not mental representations are themselves conceived as reducible to the underlying neural activity, this psychologism assumes the guise of either a materialist reductionism or a mentalist functionalism.

J-P. Changeux defends a variant of materialist reductionism. His aim is twofold: first, to inquire into the nature of mathematics, but also, at a more strategic level, to put mathematics in its place, so to speak. He has never concealed his opposition to Cartesian or Leibnizian rationalisms that have made mathematics the "queen" of the sciences. In his view, mathematics must abdicate its overly arrogant sovereignty, stop laying claim to universal validity and absolute truth, and accept the humbler role assigned to it by Bacon and Diderot—that of "servant" to the natural sciences (p. 7). And what better way to make mathematics surrender its prestigious seniority than to demonstrate scientifically that its claims to absolute truth have no more rational basis than do those of religious faith?

Pursuing his mission with great conviction, J-P. Changeux revisits all the traditional touchstones of the empiricist, materialist and nominalist critiques of Platonist idealism in mathematics. He cites an impressive mass of scientific data along the way, including results from neurobiology and cognitive psychology in which he has played a leading role. It is this aspect of his approach which commands attention.

1. The empiricist and constructivist theses hold that mathematical objects are "creatures of reason" whose reality is purely cerebral (p. 11). They are representations, that is mental objects that exist materially in the brain, and "corresponding to physical [i.e. neural] states" (p. 14).

> "Mental representations – memory objects – are coded in the brain as forms in the Gestalt sense, and stored in the neurons and synapses, despite significant variability in synaptic efficacy" (p. 128).

Their object-contents are reflexively analyzable and their properties can be clarified axiomatically. But that is only possible because, as mental representations, they are endowed with a material reality (pp. 11-15). What's more, the axiomatic method of analysis is itself a "cerebral process" (p. 30).

2. One might try to salvage an autonomy for the formal logical and mathematical levels by admitting, in line with the functionalist theses of computational mentalism in the style of Johnson-Laird, Fodor and Pylyshyn, that the algorithms of psychological "softwares" are independent of the neural "hardware" that implements them: mental representations would then constitute, as they do for Fodor, an "internal language of thought" possessing all the characteristics of a formal language (symbols, symbolic expressions, inference rules, etc.). But, according to Changeux, such theses run into a "real epistemological obstacle" because they assume that:

> "it's possible to identify a mathematical algorithm with a physical property of the brain" (p. 167).

The brain cannot be a biological computer because,

> "both the brain's program and machine […] exhibit from the first stages of development a very intricate interplay" (p. 168).

In that sense, the brain is an evolutionary Darwinian machine.

3. Even though they can be identified with mental processes and representations, mathematical objects, structures and theories are not of a purely private and subjective nature. That would lead to solipsism. They are also communicable, public, historical and cultural representations and, for this reason, "secular" and "contingent" (p. 18). They are selected by a contingent evolutionary process. They are

> "cultural objects, (...) public representations of mental objects of a particular type that are produced in the brains of mathematicians and are propagated from one brain to another" (p. 35).

Mathematics constitutes *a language* and must therefore be approached cognitively, like any other language, taking off from cognitive theories of concept formation, abstraction, symbolic coding, reasoning, procedures, learning, etc. It follows that there can be no *ontology* of mathematics: here evolutionist historicism (with chance becoming "necessity" through selection) takes the place of ontological necessity. The reality, existence, coherence, and rigidity of mathematics are "a posteriori results of evolution" (p. 36).

> "The science of the 'why?' isn't theology, it's evolutionary biology. And the 'why?' of the existence of mathematics has as much to do with the evolution of our knowledge acquisition apparatus – our brain – as it does with the evolution of mathematical objects themselves" (p. 40).

4. Obviously, there exist several different levels of cognitive organization, from the most concrete (the perceptive) to the most abstract (the symbolic). They are realized in the neural architecture, from elementary neural circuits of the spinal cord, the brain stem and ganglions (p. 98) all the way to the frontal cortex, the seat of "the neural architectures of reasoni" (p. 104), not to mention the neural assemblies that code the cognitive acts of "understanding" (what is called "population coding"). This hierarchical complexity, of which we are beginning to get a pretty good grasp, obviously plays a fundamental role in the progressive structuring of the mathematical universe.

5. The evolutionist conception of mathematical epistemology leads to a "mental Darwinism" which J-P. Changeux develops in detail as a "new idea". This idea is, let it be said once again, that the brain is a natural evolutionary machine

> "[that] evolves in a Darwinian fashion, simultaneously at several different levels and on several different time scales" (p. 168).

The general model of Darwinism combines, as we know, a generator of diversity with a system for selection. At a certain level of organization (itself rooted in lower levels), elements functioning as "matter" combine to generate the "forms" ("Darwinian variations") of the next level. Some of these forms are stabilized through selection on the basis of their functional efficacy. In this sense,

> "the function feeds back into the 'variation-form' transition" (p. 108).

Changeux was one of the first, along with Antoine Danchin and Philippe Courrège, to propose a detailed model of the fundamental mechanism of *epigenesis* through *selective stabilization of synapses*. This explains how neural Darwinism naturally extends into a psychological Darwinism pertaining to the generation/selection of representations.

6. This purely representational and communicational, cognitive, neural and Darwinian reality of mathematical activity is then used to justify a materialism denouncing any Platonism as an irrational belief. The Platonic realism which holds that

> "mathematical objects exist 'somewhere in the universe', independently of all material and cerebral support" (p. 18)

is, according to Changeux, the "mythic residue" (p. 25) of a bygone magico-theological age, an irrational belief that must be eliminated through the "intellectual ascesis of the materialist" (p. 25). The materialist epistemology which, since Galileo, has been the "victim of a special form of intolerance" (p. 26) is

> "the best one available to the informed scientist [who is honest with himself]" (p. 26).

Mathematical objects cannot exist in nature. They are not natural objects. Where then could they exist? For "to exist" means, and can only mean, to exist in nature, to "exist in the universe prior to [their] existing in the brain of the mathematician" (p. 41), in short, to exist materially as an independent substance outside the mind. Mathematics can therefore be nothing but a series of mental constructions. Which, if Changeux is to be believed, is what Kant already said:

> "the ultimate truth of mathematics lies in the possibility of its concepts being constructed by the human mind" (p. 40).

Thanks to a subtle rhetoric, Galileo, who was condemned for having elevated mathematics to the rank of an objective essence of natural reality, and Kant, who never ceased to assert the absolutely irreducible role of pure mathematics in constituting the objectivity of the true sciences, thus find themselves enlisted in the service of an anti-mathematical materialism.

7. Such a conception of the reality of mathematical idealities obviously leads to an "extremely concrete and pragmatic" (p. 64) conception of their applicability. Mathematics is not the "organizing principle of matter." It is only "a rough language" for *describing* the latter. To be sure, there exist regularities in nature, but these are "properties intrinsic to matter" and not mathematical laws (p. 46). Mathematics confines itself—and should confine itself—to providing models (foreign to nature), which are selected by the scientific community on the basis of what "fits the best with the real world" (p. 64). Moreover, as several examples demonstrate, a mathematical equation (such as that of Hodgkin and Huxley for the nerve impulse, for example)

> "describes a function. It allows us to grasp a certain behavior, but not to fully *explain* the phenomenon" (p. 60).

An explanation would require the identification of the underlying structure (in the case at hand, the biological structure of the channels for sodium and potassium ions in the axon membrane) (p. 60).

But the argument here is hardly self-evident. The mathematical models of physical theories always operate at a certain level of reality. The Navier-Stokes equations are called upon to determine the flow dynamics of liquids and not their molecular structure which, for its part, will be marvelously described by the equations of quantum mechanics. The equations of Newton and Einstein are called upon to determine gravitational interactions and not the chemistry of planets, etc. As to the explanation of a phenomenon by underlying structures, it is clearly no longer possible at the level of fundamental physics even though this level universally constrains all the other levels of reality.

8. Given this set of "self-evident truths", epistemologists who refuse to identify the "apparatus of knowledge" and the brain can only do so through their "ignorance of neuroscience" (p. 25). Here Jean-Pierre Changeux takes explicit aim at Jean-Toussaint Desanti, the leading French philosopher of mathematics of the post-war era, who, in his authoritative work *Les idéalités mathématiques*, took up and developped many important theses first advanced by Edmund Husserl and by Jean Cavaillès.

II. ALAIN CONNES' STRUCTURAL OBJECTIVISM

Evolutionist biological materialism and neural Darwinism are certainly positions with a great deal of validity. They should ultimately lead to a complete rethinking of the foundational problems of mathematics. If we may be allowed to offer a bit of personal testimony, we are ourselves involved in theorizing the neural bases of space using models of the functional architecture of visual areas and of the kinesthetic coupling of perception and action, and we have witnessed the extent to which the question of the foundations of geometry is thereby transformed (see Petitot [2003]). But, for all that, neural Darwinism does not make it possible to "psychologize" mathematics.

Indeed, a classic difficulty encountered by reductionist materialism derives from the fact that it identifies *objects* with the cognitive *acts* that provide access to them. It maintains that mathematical idealities cannot exist because:

- existence is equivalent to a sort of ontological independence, what philosophers call "transcendence", and
- no ontological transcendence could arise out of the immanence of cognitive acts.

The conclusion reached from these premises is that the reality content of mathematical objects must be reduced to the conditions of epistemic access to them. But Alain Connes rejects this conclusion outright, for, in his view, mathematical reality is fundamentally different

> "from the manner in which it is apprehended" (p. 14).

The materialist thesis is of course perfectly defensible. But if he adopts it, a "well-informed scientist who is honest with himself" must adopt it *fully* and accept all the consequences of his refusal of any realism where abstract entities are concerned. As it happens, ever since the medieval controversies between realists and nominalists over this question, which is nothing other than the question of "universals", humanity has devoted a great deal of reflection to the matter. Now, among all the consequences, there is one in particular, traditional but formidable, which, from Berkeley to Husserl and Quine, philosophers have analyzed in all its facets. Materialism, and the nominalism that goes with it, presuppose an independent reality composed of individualized and "separate" substances. But how do we obtain access to this transcendent material reality? Through the objects of our perception (aided by all the measuring devices one might want), that is, through *phenomena.* But the phenomena which are the objects of perception are a prototype of cognitive construction. They are *constituted* out of sensory data and, insofar as they are constituted, they are just as *abstract* as pure numbers.

In other words, the anti-realist thesis concerning mathematics must then be extended to perception itself and that leads, with no hope of escape, to a radical solipsism. If one adopts an ontological realism with respect to external material reality, one is of course able to justify an anti-Platonism where mathematical entities are concerned, but one finds oneself equally obliged, reluctantly but ineluctably, to reject the reality of perception and thus to *invert* ontological realism into subjective idealism.

Hilary Putnam has studied very discerningly the conflict between physicalist realism and commonsense realism which runs through our modern conception of reality. In his 1987 essay *The Many Faces of Realism* (The Paul Carus Lectures), he recalls the genesis of the dualism between, on the one hand, the ontology of a transcendent, independent external reality existing in itself and, on the other hand, the cognitive reconstruction of the perceived world through sensory data, and he shows to what extent this dualism is detrimental.

For, once they cannot be expressed in the language of physics, how are we to think about the qualitative structures of the phenomenologically manifested world? According to Putnam, we must call into question the commonly accepted opposition between properties that are intrinsic (i.e. transcendent and independent of the mind, of perception, and of language) and properties that are extrinsic, apparent, projected and dispositional. As he puts it:

> "to explain the features of the commonsense world, including color, solidity, causality […] in terms of a mental operation called 'projection' is to explain just about every feature of the common-sense world in terms of thought" (p. 12).

The immediate result is that, in practice, realism reverses itself into a pure subjective idealism:

> "So far as the commonsense world is concerned […] the effect of what is called 'realism' in philosophy is to deny objective reality, to make it all simply thought" (p. 12).

Putnam goes on to explain that if one wishes to develop a physicalist monism on these bases, one is obliged to interpret mental phenomena as complex and derived physical phenomena. But, as the theses of *functionalism* make explicit, there is no necessary and sufficient condition (NSC) characterizing mental contents and propositional attitudes that can be formulated in physical language. Such an NSC would in fact be infinite and lack effective rules of construction. The intentionality of consciousness remains, it seems, irreducible to the physical and the computational levels. But then, it should itself be conceived as a projection, which is absurd.

In his debate with Jean-Pierre Changeux, Alain Connes has made a very good case for this point. Apart from an irrational belief in the reality of the external material world, what *proves* this reality if not the *coherence* of perceptions? If mathematics were reduced to nothing but a language and if one denied any reality to it, then there would be no reason not to deem perceived real objects to be merely

> "a mental construction useful for explaining certain visual phenomena" (p. 23).

That is why

> "reducing [mathematics] to a mere language would be a serious mistake" (p. 22).

If mathematics is effectively reduced to the brain, why then not equally reduce the world to the brain through the intermediary of perception? (p. 56). In the debate, J-P. Changeux rejects this parallel between mathematics and perception as a "metaphor." But the argument carries considerable weight. It can even be reinforced by applying it not only to the objects of perception but to those physical objects which themselves constitute, for the materialist, the ultimate ontological reality. In this sense, the argument has been spelled out quite well by Quine.

Quine has remarked that the physical objects postulated by physical theories are neither more nor less ideal than mathematical idealities themselves and that it is therefore just as legitimate, or just as illegitimate, to accept the former as it is to accept the latter. One cannot be, at one and the same time, a realist in physics and a nominalist in mathematics. Physical objects too are explanatory idealities that allow us to reduce the complexity of sensory experience to a conceptual simplicity.

> "Platonist ontology […] is, from the point of view of the strictly physicalistic conceptual scheme, as much a myth as that physicalistic conceptual scheme itself is for phenomenalism." (Quine [1948])

As soon as one treats physical objects as real, one must accept their existence ("ontological commitment"). But then one must equally accept the existence of mathematical idealities. One's ontological commitment must be *coherent*. The refusal to do so would amount to "intellectual dishonesty" (see Maddy [1989], p. 1131; it will be noted that both Changeux and Quine appeal to the intellectual "honesty" of their peers). Consequently, Quine criticizes the positivists who seek to exclude as non-sensical statements on the existence of abstract objects. Mathematics is part of science and

> "we can have reasons, and essentially scientific reasons, for including numbers or classes or the like in the range of values of our variables" (Quine [1969], p. 97).

For this debate to move forward, it is philosophically necessary to change viewpoints and to realize that the problem is not that of an *ontology* of mathematics in the traditional sense, but rather that of its *objective reality*. Alain Connes clearly positions himself on this terrain when he insists upon the reality of mathematical idealities, for example, in the case of prime numbers, the infinity of which is

> "a reality every bit as incontestable as physical reality" (p. 13).

> "prime numbers (...) constitute a more stable reality than the material reality that surround us" (p. 12).

A. Connes returns several times to the necessity of admitting such a mathematical reality as "raw and immutable" and not reducible to the conceptual tools employed to investigate it, a reality

> "every bit as constraining as physical reality, but one that's far more stable than physical reality, for it is not being located in space-time" (p. 28).

No serious philosophical debate about modern science is possible if one fails to distinguish carefully *between ontology and objectivity*. But once one has done so, thus posing the problem of the reality of mathematical idealities in terms of their objective status rather than in terms of an ontology, existence in the spatio-temporal world is no longer the exclusive criterion of reality and it becomes possible to display *criteria* of objectivity. Alain Connes repeatedly underscores three such criteria, which are indeed absolutely fundamental.

1. The possibility of exhaustively classifying the objects defined by an axiomatics, the axiomatics allowing

> "classification problems to be posed for mathematical objects defined by very simple conditions" (p. 13).

For example, for every prime number p and every positive integer n there exists one and only one finite field of characteristic p and of cardinal p^n and one thus obtains all the finite fields (p. 13). The complete classification of locally compact fields is equally known (the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers, the p-adic fields and their algebraic extensions, the fields of formal series over finite fields, p. 16). In the same way, an uninterrupted series of brilliant efforts (from Galois to Chevalley and then Feit and Thompson) have led to the classification of simple finite groups. One could cite many other examples from topology, geometry, etc. This history begins with the Greek geometers who classified the five Platonic solids. Such results manifest the existence of objective constraints that necessarily limit the domains of possibility.

2. The *global* inter-theoretical *consistency and harmony* of mathematical theories (p. 152). Despite being "inexplicable" (p. 17) and constituting a crucial problem, these are incontestable and objective. They are "the very opposite of randomness" (p. 116). This aspect of things cannot be overemphasized. As Jean Dieudonné has observed with respect to what the great philosopher Albert Lautman called the *unity* of mathematics, all the major theorems bring into play a huge number of different theories and manifest absolutely unsuspected solidarities among apparently unrelated objects and structures. Among the examples supplied by A. Connes, one might single out the way in which V. Jones, working in analysis on the classification of the "factors" of Von Neumann's algebras, used a braid group in one of his proofs and, when making the link with knot theory, discovered a new invariant which, since then, has proved to have fundamental applications in quantum field theory. One could cite a significant number of other examples which have brought to light unforseeable overall solidarities between apparently quite distant areas of mathematics and which have had remarkable physical applications. This holistic consistency is quite astonishing and shows that

> "mathematical reality, in its very structure, its internal harmony, is an inexhaustible source of organization" (p. 125).

The "immediate comprehension" of it on the part of mathematicians is essential to their creativity and to their understanding of the power of new tools (p. 152). But it remains quite enigmatic.

3. The fact that interesting mathematical theories *possess an infinite informational content*. Gödel's incompleteness theorem

> "in its most profound formulation […] shows that mathematics can't be reduced to a formal language" (p. 159).

It means that interesting structures (able to code arithmetic) contain an infinite quantity of information that cannot be finitely axiomatized. As it is explained in the French edition (p. 213) in reference to Chaitin's works:

> "One may consider this theorem to be a consequence of constraints imposed by the theory of information, due to the finiteness of the complexity of any formal system".

From this a criterion of objectivity may be deduced, for

> "isn't *that* the distinguishing characteristic of a reality independent of all human creation?" (p. 160).

It will be noticed that these criteria of objectivity are not satisfied by any other cognitive symbolic system, whether one thinks of natural languages or of the various "games" (chess and other systems of rules) to which mathematics has been compared. To be able to see them for what they are — "to do justice to what we know" — a correct doctrine of objectivity is called for.

III. THE ANTINOMY OF MATHEMATICAL REALITY

The anti-Platonic theses, whether they be psychologistic, empiricist, nominalist and/or materialist, or neurocognitive (the repertory is rather vast), seem at first sight to be self-evident. However, they are not nearly as self-evident as they look. There are several reasons for this.

1. First, they all rest upon a certain *preconception* of what physical objectivity is. They conceive external reality as founded on a substantialist ontology of autonomous material things (independent of the mind, transcendent) endowed with a sufficient structural stability and maintaining relations of causality (material and efficient) and reciprocal interaction between themselves. What's more, this substantialist ontology is admitted to be, if not explicable, at least describable by an appropriate scientific language of description built on natural language. Different levels of organization are then introduced and it is posited (reductionist thesis) that the lower levels causally explain the higher levels. Atoms, molecules, the genome, proteins, neurotransmettors, etc. really and truly exist in nature, while mathematical structures such as numbers are not supposed to exist in the same manner and will be conceived as the product of a contingent symbolic, historical and cultural evolution.

Jean-Pierre Changeux vigorously and rigorously defends a materialism of this type and he does so in a consistent fashion that does not suffer from the kind of inconsistency denounced by Quine. He doubts that numbers exist:

> "I have a hard time (...) imagining that integers exist in nature" (p. 28)

but he is just as dubious about the constructs of theoretical physics:

> "atoms exist in nature – but Bohr's atom doesn't" (p. 28).

In this conception, basic material reality functions metaphysically as a reality in-itself. Now, the hypothesis of a material reality existing in itself, transcendent and independent—and, moreover, of an independent reality satisfying a substantialist ontology of things—is a hypothesis which is *itself anti-scientific* and equally based on an irrational belief.

Not that the rational idea of such a reality in-itself should be rejected. One might well hypothesize that it "exists" as a transcendent "foundation" of empirical reality. The problem is that, as it can be argued since Kant, this foundation is cognitively inaccessible and therefore cannot be used in scientific reasoning.

What meet here an inescapable scientific datum: physics does not describe a substantial world of structurally stable material things, interrelated and interacting in causal fashion. At the fundamental (quantum) level, physical phenomena are devoid of any underlying ontology. This is a well-known theorem (Von Neumann, Bell, Kochen-Specker). In the very technicity of their physico-mathematical contents, the fundamental physical theories (symplectic mechanics, general relativity, quantum field theory and Feynman's integrals, gauge theory, string theory, etc.) confirm that objectivity cannot be identified with an ontology.

It must be said that here again Changeux is perfectly consistent. In the exchanges on quantum mechanics, he defends the principle of theories with hidden parameters. To his mind, quantum theory "is bad" because it rests on presuppositions that do not satisfy the principle that

> "the experimental conditions must be defined in such a way that it [the quantum phenomenon in question] becomes reproducible" (p. 71).

In other words, quantum mechanics is incomplete and

> "there remains an unexplained sublevel to which theoreticians haven't yet gained mental access" (p. 71).

As we know, however, it is contradictory to try to "complete" quantum mechanics preserving the locality of interactions.

2. One of the aspects of this problem concerns the way in which the anti-Platonist materialist viewpoints under discussion use in a non-problematized fashion certain concepts which, however, are fundamentally problematic. We will confine ourselves here to citing the simple but absolutely crucial concept of *space-time* and that of *continuum* which undergirds it.

Space-time is not in itself a physical reality with which we can enter into causal interactions. As Kant was the first to explain, in his celebrated "exposition" of the Transcendental Aesthetics, it is a *form* of external reality. If to exist means to exist materially in nature, then space-time *does not exist* in this sense. It, too, is, like mathematical idealities, a pure mental representation. Which, by the way, fits well with the hypothesis that its mathematical (geometric) determination should also be of an exclusively mental nature. There is a catch, however. It follows that the substantialist ontology serving as the foundation of materialist positivism should then quite logically be, as in Leibniz, an *a-spatial* and *a-temporal* ontology. The problem is that the physical objectivity to which recourse is constantly made as the materialist foundation is in the last resort entirely constructed on a spatio-temporal basis. For "to exist" is taken to mean "to exist in nature" and "to exist in nature" is taken to mean "to exist in space-time." It is a recurrent paradox of materialisms and nominalisms that they refuse the reality of abstract entities in order to confine ontology into independent, individual and "separate" substances while simultaneously subordinating this very ontology to a space and a time which are prototypical instances of idealities, fully as cognitive and abstract as numbers, and which, therefore, do not exist…

To this it must be added that space and time are based on the continuum and that the latter may be "arithmetized," in other words, reduced to numbers (even if that raises very difficult questions, as we shall see).

And this paradox will be taken to dizzying heights by modern physics because, in its physical determinations, matter is fundamentally identified – since Riemann and Clifford – with a *geometry*, or, more precisely, to borrow Wheeler's term, with a "geometro-dynamics" founded on the geometry of space-time. From general relativity to contemporary gauge theories, to super-string theory and to Alain Connes' work on the physical applications of non-commutative geometry, all of modern physics confirms Clifford's slogan "Physics is Geometry." Now, space-time is not an independent reality in itself. It is devoid of any ontological content. And yet, if one uses this fact to justify reducing it to a mere appearance, a mental projection, one will be inexorably condemned to adopt a solipsistic idealism.

To get out of this dilemma, one needs to understand that space-time is objective and not ontological – that it is in fact the primary form of physical objectivity. As Kant said, one must succeed in maintaining at one and the same time the thesis of the "empirical realism" of space *and* that of its "transcendental ideality." But once objectivity has been distinguished from ontology, there is no longer any reason to deny mathematics the same objective status as physics – quite the contrary.

IV. MATHEMATICAL IDEALITIES AND OBJECTIVITY

Jean-Pierre Changeux's point of view will no doubt be accepted and defended by a majority of scientists. It is part of the current revival of "psychologism" powerfully fueled by the various schools of epistemology which seek to "naturalize" the problems of the theory of knowledge by reducing them to problems of a cognitive psychology founded on the neurosciences. The majority of these schools are obliged to deny any reality to mathematical objects, structures and theories for the following quite obvious reason: if to exist objectively means to exist materially in nature, then how can one obtain epistemic access (learning, beliefs, knowledge) to external abstract entities *with no causal efficacy*? As Michael Resnik puts it so well:

> "if we have no physical traffic with the most basic mathematical entities and they are not literally the products of our own minds either, how can we learn any mathematics? How could it even be possible for us to acquire beliefs about mathematical objects?" (Resnik [1988], p. 403).

To salvage such a problematical mathematical ontology, one then must always introduce, in one way or another, "supernatural" cognitive faculties on the order of an intellectual intuition (cf. Frege, Husserl with his "intuition of essences", Gödel, etc.). Since that is clearly incompatible with a naturalized epistemology, there is no choice but to fall back onto a materialist nominalism.

This last point is essential. A fundamental thesis, linked to what is called the causal theory of reference, is that no knowledge of and no reference to external abstract entities can be legitimately introduced and used insofar as all knowledge of and all reference to an external entity requires a causal interaction of the subject with that entity. Now, by definition, an abstract entity cannot sustain causal relations. As Philip Kitcher asserts, it is therefore impossible for symbolic constructions and manipulations to

> "provide any type of access to abstract reality." (Kitcher [1988], p. 527)

Mathematics must be conceived, on the contrary, as a symbolic activity of a logico-linguistic (and even semio-narrative: "in certain respects, mathematics is like story-telling") nature which allows us, through a series of successive approximations sedimented in the traditions, to structure our experience more and more adequately by means of idealities. Mathematics will have emerged, through a process of idealization, from *proto*-mathematical (perceptive, etc.) knowledge constrained by the structures of natural reality. Transmitted historically and socially through the scientifico-technical legacy of humanity, it will have progressed in the same way as all of humanity's other symbolic formations. It can therefore be understood without there being the least need to invoke a mysterious world of ideas to which an incomprehensible intellectual intuition would grant us access.

Of course, the whole problem with such a line of reasoning is that it presupposes that we know the meaning of terms such as "external reality," "matter," "causality," etc. But it is impossible to define these terms objectively except in mathematical fashion. And that is precisely where the difficulty lies. The belief in the possibility of understanding the concept of "reality" independently of any objective determination and constitution is a belief even more irrational and archaic than naïve Platonism.

It can thus be seen that one's conception of the *reality* of mathematical idealities is tightly bound up with one's conception of their *applicability*: the fundamental physical objects are themselves mathematical constructions in the first place. Hence the question which for M. Resnik is one of the most important ones in the philosophy of mathematics:

> "How can we retain the advantages of an ontology of abstract entities for mathematics while removing its obvious epistemological disadvantages?" (Resnik [1988], p. 407)

The problem is clear. If, as the nominalists insist (see Hartry Field, for example, in *Science Without Numbers* [1980]), there do not exist mathematical idealities possessing the status of things, then what are the "truth-makers" for mathematical statements? It is consistent to posit, with the second Wittgenstein, that mathematical contents are *prescriptive* and not descriptive – that they are nothing but rules for the use of concepts. But as soon as one abandons this radical position, then the problem of truth-makers becomes crucial once more. As Crispin Wright recalls,

> "the traditional platonist answer is that the truth-conditions of pure mathematical statements are constituted by the properties of certain mind-independent abstract objects, the proper objects of mathematical reflection and study." (Wright [1988], p. 426)

Other answers are well known. For classical intuitionists, mathematical statements refer to mental constructions that have to be investigated with a particular logic, reflecting their constructive character (but, as we know, thanks in particular to the work of F.W. Lawvere and M. Tierney, the intuitionist logic is the internal logic of universes of sets endowed with a certain structure, and in particular of Grothendieck's topoï that is, categories of sheaves over categories endowed with a "topology"). For formalist structuralists, mathematical statements refer to structures, etc.

These questions belonging to the pure philosophy of mathematics enter into the Changeux-Connes debate with regard to the opposition between formalism and constructivism. Jean-Pierre Changeux rightly emphasizes that many intuitionist and constructivist philosophers of mathematics agree with him on anti-Platonism. He refers in particular to Allan Calder's denunciation of realism. Alain Connes picks up the argument by underscoring the fact that

> "the distinction between constructivism and formalism is a methodological distinction more than anything else" (p. 42).

and by discussing the example of the axiom of choice (AC), which is the prototype of a non-constructive axiom whose consequences are omnipresent in proofs.

If we may be allowed to express a personal opinion here, I would say that the non-constructive axioms of existence in mathematics must indeed be understood as methodological principles whose only value lies in their operational capacity. But in mathematics "methodological" means a lot because *the object is in this case the correlate of the method*. The theory of the continuum provides an especially striking example (See Petitot [1995]). Let us introduce it in the debate.

A "good" theory of the continuum consists in showing that large classes of subsets of the field ***R*** of real numbers are "regular" in the sense of sharing "good" properties, such as "being measurable in Lebesgue's sense" or "possessing the perfect set property". Cantor had already shown that the closed subsets of ***R*** are "regular" in this sense, and it was subsequently shown that it is also the case for the hierarchy of Borel subsets obtained from open and closed subsets by countable union and intersection and complementation. But there exists a more complex hierarchy, called the *projective* hierarchy, which brings into play somewhat more complex principles of construction. The projective subsets $\Sigma^1{}_n$, $\Pi^1{}_n$, $\Delta^1{}_n = \Sigma^1{}_n \cap \Pi^1{}_n$ are obtained from open subsets by iterating the set theoretical operations of complementation, countable union and projection (direct image through continuous application). It can be shown that the smallest projective class $\Delta^1{}_1$ is the class of all Borel subsets (Suslin's theorem). With respect to these new classes it is also natural to raise the question of their regularity. But such a proof quickly becomes impossible in ZFC (the standard set theory of Zermelo-Fraenkel with AC), starting with the $\Sigma^1{}_2$ and $\Pi^1{}_2$ levels, in fact for *meta-mathematical* reasons pertaining to Gödel's incompleteness theorem .

Hence Gödel's idea of completing the ZFC axioms. Gödel began by introducing the constructive theory of sets where all sets are "constructible." But the constructibility axiom turns out to be too constraining. It entails the existence of a low-level ($\Delta^1{}_2$) projective well ordering of ***R*** and thus the existence of a non-measurable Lebesgue $\Delta^1{}_2$ set . Now, a well ordering of ***R*** should be highly non-constructible and undefinable. Generally speaking, the axiom of choice AC (which remains true in the universe of constructible sets) entails the existence of very complicated and very irregular sets that are nonetheless projective. These sets should be highly non-constructive. But the axiom of constructibility forces them to exist in the hierarchy of projective sets. Hence a complete reversal of strategy on Gödel's part.

Priority is now given to being able to prove good properties of regularity for projective sets and to generalizing the results of Luzin and Suslin proving that every Σ^1_1 and every Π^1_1 are Lebesgue measurable and that every Σ^1_1 displays the perfect set property. This involves enriching the axioms of set theory by specifying the size of the universe. The best way of doing this is to introduce new existence axioms known as *large cardinals* axioms (inaccessible, measurable, etc.) which introduce higher levels of infinity into the transfinite. From the standpoint of large cardinals axioms it is no longer a matter of elaborating a model of the continuum that is dogmatically constrained by a constructivist *a priori*, but rather of reconstructing as well as possible, from within mathematics itself, the "intuitive" continuum.

The fundamental result is then that the "good" structure of the continuum in a ZFC universe is the *counterpart* of very strong non-constructible (Platonist) axioms of existence for large cardinals. We may therefore consider these, as Gödel and Martin proposed, to be hypotheses regarding, not a fixed and completely predetermined mathematical universe, but a universe to be determined in the most harmonious way possible.

We see that, if one wishes to avoid a cascade of insurmountable difficulties, one must not apply to mathematics, apart from the relationship between syntax and semantics proper to the logical theory of models, the traditional and general conception of a denotative relationship between a language and a reality (theory of reference). Indeed, it is only at this point that one runs into the problem of what makes mathematical statements true (in the sense of a truth-correspondence) and of what allows us to know that true statements are true (epistemic accessibility to truth).

The conception of denotation and of truth that one adopts will determine how one conceives the nature of proofs. For a traditional Platonist, proofs are only cognitive auxiliaries providing access to independent truths (with ontological content). In this perspective, truth thus transcends provability. Radical finitist intuitionists like Wittgenstein and Dummett deny this thesis: for them, a mathematical truth cannot transcend the proof that determines it. But then the problem of the applicability of mathematics becomes incomprehensible. For, as Crispin Wright emphasizes:

> "How is it possible to apply mathematics to statements which concern ordinary things, and how does the credibility which attaches to a pure mathematical statement as a result of proof carry over to its application?" (Wright [1988], p. 429)

Personally, we believe that the question of the reality and the applicability of mathematical idealities should not be conceived in terms of an analogy with the relationship between a language and the world. In their relationship to reality, mathematical theories do not denote, any more than do the physical theories which bring them into play. They *determine* – they legalize – phenomenal data, which is something else altogether. To be sure, the theory of models internalizes, in meta-mathematics, a relationship which is apparently of the "language-reality" type. But the latter is intra-mathematical and thus does not entail any relationship with an external world. Consequently, it remains foreign to the questions of reality and of applicability. To pose these questions while trying to couple this meta-mathematical relationship with an "ontological" relationship of the "(mathematical) language - (real) world" type amounts to conceiving knowledge in terms of predication and denying the essential gap separating science from common sense. To conceive of knowledge in terms of predication is to cling to a classical metaphysical tradition that no longer possesses any value. It means neglecting the philosophical fact of its having been replaced by a problematics of objectivity.

The problem of the reality of mathematical idealities is not that of their reality in a traditional ontological sense, but that of their objectivity, which is – one cannot emphasize this enough – something else entirely. The notion of reality is a *modal category* inseparable from a transcendental doctrine of constitution and not an absolute concept. Likewise, the problem of the applicability of mathematical idealities is not that of their applicability to an ontological reality of the world, but that of their entailment in physical objectivity, which, once again, is something else altogether.

CONCLUSION

The debate between Jean-Pierre Changeux and Alain Connes is one of the most interesting to take place in recent years. It re-frames in a very up-to-date context a whole series of traditional and difficult questions from the standpoint of the knowledge and experience of two of the leading protagonists of contemporary science. To the choice developed by the neurobiologist between a Platonist ontology and a neurocognitive psychology of mathematical activity, the mathematician replies with a conception that is objective (neither ontological nor psychological) of the thoroughly consistent universe of mathematical idealities. It is indeed in this three-sided arena that the major difficulties play themselves out. One of the great virtues of the book is to cast a spotlight on this confrontation.

Translated by Mark Anspach

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