Carathéodory interpolation problem over quaternions and related questions

Vladimir Bolotnikov

GAG seminar at W&M

The problem consists of finding a function from a given functional class with prescribed Taylor coefficients about a given point.

The classical versions are concerned about functions f analytic in the open unit disk $\mathbb{D}$ and such that $|f(z)| < 1$ (the Schur class $\mathcal{S}$) or $\Re f(z| > 0$ (the Carathéodory class $\mathcal{C}$).

C. Carathéodory (1907) described the set of all points $(c_1, \ldots, c_n) \in \mathbb{C}^n$ such that there exists

$$f(z) = 1 + c_1 z + \ldots + c_n z^n + \ldots \in \mathcal{C}$$

as the closed convex body in $\mathbb{C}^n$ whose boundary points correspond to rational functions

$$\sum_{j=1}^{n} \gamma_j \cdot \frac{\lambda_j + z}{\lambda_j - z} \quad (\gamma_j \geq 0, \ \ |\lambda_j| = 1)$$

taking purely imaginary values everywhere on the unit circle except for at most n simple poles.

C. Carathéodory (1911, Rendiconti del Circolo Matematico di Palermo) Let $E \subset \mathbb{R}^n$. Then any $x \in E$ in the convex hull of E is a convex combination of at most $n+1$ points from E.

O. Toeplitz (1911): there exists

$$f(z) = c_0 + c_1 z + \ldots + c_{n-1} z^{n-1} + \ldots \in \mathcal{C} \qquad (1)$$

if and only if $Q_n = \begin{bmatrix} c_0 + \overline{c}_0 & \overline{c}_1 & \ldots & \overline{c}_{n-1} \\ c_1 & c_0 + \overline{c}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{c}_1 \\ c_{n-1} & \ldots & c_1 & c_0 + \overline{c}_0 \end{bmatrix} \succeq 0$

To construct a concrete $f \in \mathcal{C}$ as in (1), it suffices to extend the given $\{c_j\}_{j=0}^{n-1}$ to an infinite positive-definite sequence $\{c_j\}_{j=0}^{\infty}$ (such that $Q_k \succeq 0$ for $k \geq n$). Such an extension always exists and is unique if and only if Q_n is singular. In this case, $\mathrm{rank} Q_k = \mathrm{rank} Q_n$ for all $k \geq n$.

Another question: to describe all $f \in \mathcal{C}$ as in (1) or all positive-definite extensions of $\{c_j\}_{j=0}^{n-1}$ in the indeterminate case.

Let

$$\mathbf{T}_n^f = \begin{bmatrix} f_0 & 0 & \dots & 0 \\ f_1 & f_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{n-1} & \dots & f_1 & f_0 \end{bmatrix} \quad \text{if} \quad f(z) = \sum_{k=0}^{\infty} f_k z^k.$$

Thus, $f \in \mathcal{C}$ if and only if $\mathbf{T}_k^f + \mathbf{T}_k^{f*} \succeq 0$ for all $k \geq 0$ and either all these matrices are invertible or

$$\operatorname{rank}\left(\mathbf{T}_k^f + \mathbf{T}_k^{f*}\right) = \min(k, n) \quad \text{for some} \quad n \geq 0.$$

In the latter case, f is a rational function as on the page 1.

I. Schur (1917): given $f_0, \ldots, f_{n-1}$, there exists

$$f(z) = f_0 + f_1 z + \ldots + f_{n-1} z^{n-1} + \ldots \in \mathcal{S} \tag{2}$$

if and only if $P_n = I - \mathbf{T}_n^f \mathbf{T}_n^{f*} \succeq 0$ (i.e., $\mathbf{T}_n$ is a contraction). In this case $\{f_j\}_{j=0}^{n-1}$ extends to $\{f_j\}_{j=0}^{\infty}$ such that $P_k \succeq 0$ for $k \geq n$.

The extension is unique if and only if P_n is singular. In this case $\mathrm{rank} P_k = \mathrm{rank} P_n$ for all $k \geq n$ and the corresponding f is a finite Blaschke product, $\deg f = \mathrm{rank} P_n$.

Thus, $f \in \mathcal{S}$ if and only if $P_k = I - \mathbf{T}_k^f \mathbf{T}_k^{f*} \succeq 0$ for all $k \geq 0$ and either all these matrices are invertible or

$$\mathrm{rank}\left(I - \mathbf{T}_k^f \mathbf{T}_k^{f*}\right) = \min(k, n) \quad \text{for some} \quad n \geq 0.$$

In the latter case, f is a Blaschke product of degree n. In the indeterminate case, all f of the form (2) are described by a linear fractional formula.

Another question: If $P_n = I - \mathbf{T}_n^f \mathbf{T}_n^{f*} \not\succeq 0$, i.e., if $\nu_-(P_n) = \kappa > 0$, is it possible to extend it so that $\nu_-(P_m) = \kappa$ for all $m > n$?

M.G. Krein – H. Langer (1977): Yes, if P_n is invertible. Each such extension gives rise to a meromorphic f in $\mathbb{D}$ with κ poles and such that

$$\lim_{r \to 1^-} \sup_{|z|=r} |f(z)| \leq 1$$

or equivalently, to a function of the form

$$f = \frac{s}{b}: \quad s \in \mathcal{S}, \quad b(z) = \prod_{i=1}^{\kappa} \frac{z - a_i}{1 - z\bar{a}_i}, \quad |a_i| < 1.$$

Quaternions

$$\mathbb{H} = \{\alpha = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 : \ x_i \in \mathbb{R}, \ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\},$$

$$\Re(\alpha) = x_0, \quad \overline{\alpha} = x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3, \quad |\alpha| = \sqrt{\alpha\overline{\alpha}}.$$

Associated with any non-real α are its centralizer

$$\mathbb{C}_\alpha = \{\beta \in \mathbb{H} : \ \alpha\beta = \beta\alpha\} = \mathbf{span}_{\mathbb{R}}(1, \alpha)$$

and the similarity (conjugacy) class

$$[\alpha] := \{h\alpha h^{-1} : \ h \neq 0\} = \{\beta \in \mathbb{H} : \ \Re(\beta) = \Re(\alpha) \ \& \ |\beta| = |\alpha|\}.$$

$\alpha \sim \beta$ if and only if

$$\boldsymbol{\mu}_\alpha(z) = z^2 - 2z\Re\alpha + |\alpha|^2 = z^2 - 2z\Re\beta + |\beta|^2 = \boldsymbol{\mu}_\beta(z)$$

Matrices over quaternions

Given a matrix $A = [a_{ij}]$, its adjoint (conjugate transpose) A^* is defined as $A^* = [\bar{a}_{ji}]$. If A is *Hermitian* (i.e., $A = A^*$) all its eigenvalues are real; if they are all nonnegative (equvalently, $\mathbf{x}^* A\mathbf{x} \geq 0$ for any $\mathbf{x} \in \mathbb{H}^n$) the matrix A is called *positive semidefinite*. If all eigenvalues are positive (equivalently, $\mathbf{x}^* A\mathbf{x} > 0$ for any $\mathbf{x} \in \mathbb{H}^n$), A is called positive definite.

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Cauchy interlacing theorem: If $A \in \mathbb{H}^{n\times n}$ is a Hermitian matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$, and $B \in \mathbb{H}^{m\times m}$ is a principal submatrix of A with eigenvalues $\mu_1 \leq \ldots \leq \mu_m$, then $\lambda_k \leq \mu_k \leq \lambda_{k+n-m}$ for $k = 1, \ldots, m$.

R.C. Thompson, Johns Hopkins Lecture Series, 1988.
T.Y. Tam (1999).

Stein equations and Schur complements

Let

$$Z_n = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{bmatrix}, \quad C_n = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \in \mathbb{H}^{n\times 2}, \quad J = J^*$$

Then the Stein equation $\boxed{P_n - Z_n P_n Z_n^* = C_n J C_n^*}$ has a unique (Hermitian) solution P_n. Furthermore, $P_{n+k} = \begin{bmatrix} P_n & B_{n,k}^* \\ B_{n,k} & D_k \end{bmatrix}$.

If P_n is invertible,

$$P_{n+k} = \begin{bmatrix} I & 0 \\ B_{n,k}P_n^{-1} & I \end{bmatrix} \begin{bmatrix} P_n & 0 \\ 0 & \mathbf{S}_k \end{bmatrix} \begin{bmatrix} I & P_n^{-1}B_{n,k}^* \\ 0 & I \end{bmatrix},$$

where $\mathbf{S}_k := D_k - B_{n,k}P_n^{-1}B_{n,k}^*$ is the Schur complement of P_n.

$$\nu_\pm(P_{n+k}) = \nu_\pm(P_n) + \nu_\pm(\mathbf{S}_k), \quad \mathrm{rank} P_{n+k} = n + \mathrm{rank}\mathbf{S}_k.$$

Remark: $\mathbf{S}_k$ satisfies the Stein identity

$$\mathbf{S}_k - Z_k \mathbf{S}_k Z_k^* = C'_k J C'^*_k,$$

where $C'_k \in \mathbb{H}^{k\times 2}$ is given by

$$C'_k = \begin{bmatrix} c'_0 \\ \vdots \\ c'_{k-1} \end{bmatrix} = (I - Z_k)\begin{bmatrix} -B_{n,k}P_n^{-1} & I_k \end{bmatrix}(I - Z_{n+k})^{-1} C_{n+k}.$$

Furthermore, the top row c'_0 in C'_k is non-zero.

Remark: $\mathbf{S}_k$ satisfies the Stein identity

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where $C_k' \in \mathbb{H}^{k\times 2}$ is given by

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Furthermore, the top row c_0' in C_k' is non-zero.

This remark applies to

$$C_n = \begin{bmatrix} \mathbf{e}_n & F_n \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad F_n = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

in which case

$$P_n - Z_n P_n Z_n^* = C_n J C_n^* = \mathbf{e}_n \mathbf{e}_n^* - F_n F_n^* \quad \Leftrightarrow \quad P_n = I - \mathbf{T}_n^f \mathbf{T}_n^{f*}$$

$$\mathbf{T}^f_{n+k} = \begin{bmatrix} \mathbf{T}^f_n & 0 \\ T_{n,k} & \mathbf{T}^f_k \end{bmatrix}, \quad P_{n+k} = \begin{bmatrix} P_n & -\mathbf{T}^f_n T^*_{n,k} \\ -T_{n,k}\mathbf{T}^{f*}_n & P_k - T_{n,k}T^*_{n,k} \end{bmatrix},$$

If P_n is invertible, then its Schur complement $\mathbf{S}_k$ is subject to

$$\mathbf{S}_k - Z_k \mathbf{S}_k Z_k^* = X_k X_k^* - Y_k Y_k^*,$$

where $X_k, Y_k \in \mathbb{H}^k$ are given by the formula

$$\begin{aligned} \begin{bmatrix} X_k & Y_k \end{bmatrix} &= \begin{bmatrix} x_0 & y_0 \\ \vdots & \vdots \\ x_{k-1} & y_{k-1} \end{bmatrix} \\ &= (I - Z_k)\begin{bmatrix} T_{n,k}\mathbf{T}^{f*}_n P_n^{-1} & I_k \end{bmatrix}(I - Z_{n+k})^{-1}\begin{bmatrix} \mathbf{e}_{n+k} & F_{n+k} \end{bmatrix}. \end{aligned}$$

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Since $\begin{bmatrix} x_0 & y_0 \end{bmatrix} \neq 0$ and $\mathbf{S}_1 = |x_0|^2 - |y_0|^2$, it follows that if $\nu_-(P_{n+1}) = \nu_-(P_n)$ (i.e., $\mathbf{S}_1 \geq 0$), then $x_0 \neq 0$. Then the Toeplitz matrix $\mathbf{T}^x_k$ is invertible, and

$$(\mathbf{T}^x_k)^{-1}\mathbf{S}_k(\mathbf{T}^{x*}_k)^{-1} - Z_k(\mathbf{T}^x_k)^{-1}\mathbf{S}_k(\mathbf{T}^{x*}_k)^{-1}Z_k^* = \mathbf{e}_k\mathbf{e}_k^* - \mathcal{E}_k\mathcal{E}_k^*,$$

where $\mathcal{E}_k = \begin{bmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_{k-1} \end{bmatrix} := (\mathbf{T}^x_k)^{-1}Y_k$.

Therefore, $(\mathbf{T}_k^x)^{-1}\mathbf{S}_k(\mathbf{T}_k^{x*})^{-1} = I_k - \mathbf{T}_k^{\varepsilon}\mathbf{T}_k^{\varepsilon *}$. By the Sylvester law of inertia,

$$\nu_\pm(P_{n+k}) = \nu_\pm(P_n) + \nu_\pm(\mathbf{S}_k) = \nu_\pm(P_n) + \nu_\pm(I_k - \mathbf{T}_k^{\varepsilon}\mathbf{T}_k^{\varepsilon *})$$

Theorem: Given $f_0, \ldots, f_{N-1}$, let us assume that the matrix $P_N := I - \mathbf{T}_N^f\mathbf{T}_N^{f*}$ is singular and that P_n ($n < N$) is the maximal invertible leading principal submatrix of P_N.

1. If $\mathrm{rank}(P_N) = \mathrm{rank}(P_n) = n$, then $\nu_-(P_N) = \nu_-(P_n) := \kappa$ and for each $m \geq 1$, the extension P_{N+m} with $\nu_-(P_{N+m}) = \kappa$ is unique and satisfies $\mathrm{rank}(P_{N+m}) = n$.
2. If $\mathrm{rank}(P_N) = d > n$, then for any choice of $f_N, \ldots, f_{2N-d-1}$,

$$\nu_\pm(P_{N+j}) = \nu_\pm(P_N) + j \quad \text{for} \quad j = 1, \ldots, N-d.$$

   In particular, the matrix P_{2N-d} is invertible.

Extensions of invertible P_n with minimal negative inertia

Define matrix polynomials $\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}$ and $\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$:

$$\Psi(z) = z^n I_2 - (z-1) \begin{bmatrix} \mathbf{e}^* \\ F_n^* \end{bmatrix} (I - Z_n^*)^{-1} P_n^{-1} \mathbf{Z}_n(z) \begin{bmatrix} \mathbf{e} & -F_n \end{bmatrix},$$

$$\Theta(z) = I_2 + (z-1) \begin{bmatrix} \mathbf{e}^* \\ F_n^* \end{bmatrix} (I - zZ_n^*)^{-1} P_n^{-1} (I - Z_n)^{-1} \begin{bmatrix} \mathbf{e} & -F_n \end{bmatrix},$$

where $\mathbf{Z}_n(z) := \sum_{j=1}^{n} z^{n-j} Z_n^{j-1}$ and $(I - zZ_n^*)^{-1} = \sum_{j=0}^{n-1} z^j Z_n^{*j}$.

Then $\boxed{\Theta(z)\Psi(z) = z^n I_2}$ and

$$\begin{bmatrix} \theta_{21,0} & \theta_{22,0} \end{bmatrix} \neq 0, \quad \begin{bmatrix} \psi_{11,0} \\ \psi_{21,0} \end{bmatrix} \neq 0.$$

Claim: If the n first coefficients of $f(z) = \sum f_j z^j \in \mathbb{H}[[z]]$ are such that P_n is invertible, then

$$\begin{bmatrix} 1 & -f \end{bmatrix} \Theta = \begin{bmatrix} \theta_{11} - f\theta_{21} & \theta_{12} - f\theta_{22} \end{bmatrix} = z^n \begin{bmatrix} x & -y \end{bmatrix}, \quad (4)$$

where $y(z) = \sum_{j=0}^{\infty} y_j z^j$ and $x(z) = \sum_{j=0}^{\infty} x_j z^j$ are the power series with coefficients defined by

$$\begin{bmatrix} x_0 & y_0 \\ \vdots & \vdots \\ x_{k-1} & y_{k-1} \end{bmatrix} = (I - Z_k) \begin{bmatrix} T_{n,k} \mathbf{T}_n^{f*} P_n^{-1} & I_k \end{bmatrix} (I - Z_{n+k})^{-1} \begin{bmatrix} \mathbf{e}_{n+k} & F_{n+k} \end{bmatrix}.$$

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It turns out that $x_0 \psi_{11,0} - y_0 \psi_{21,0} \neq 0$, and (4) can be written as

$$f = (x\psi_{11} - y\psi_{21})^{-1}(y\psi_{22} - x\psi_{12}) = (\psi_{11} - \varepsilon\psi_{21})^{-1}(\varepsilon\psi_{22} - \psi_{12}),$$

where $\varepsilon = x^{-1} y$.

Theorem: Let $f_0, \ldots, f_{n-1} \in \mathbb{H}$ be such that P_n is invertible and let Ψ and Θ be the polynomials defined two pages ago. Then

1. Equality
$$(\psi_{11} - \varepsilon\psi_{21})^{-1}(\varepsilon\psi_{22} - \psi_{12}) = (\theta_{11}\varepsilon + \theta_{21})(\theta_{21}\varepsilon + \theta_{22})^{-1}$$
holds for any $\varepsilon \in \mathbb{H}[[z]]$ subject to equivalent conditions
$$\psi_{11,0} - \varepsilon_0\psi_{21,0} \neq 0 \iff \theta_{21,0}\varepsilon_0 + \theta_{22,0} \neq 0, \qquad (5)$$
which are met for all ε with $|\varepsilon_0| \leq 1$ if and only if the bottom diagonal entry in P_n^{-1} is positive.
2. An extended sequence $\{f_j\}_{j\geq 0}$ satisfies equalities
$$\nu_-(P_{n+k}) = \nu_-(P_n) \quad \text{for all} \quad k \geq 1$$
if and only if its Z-transform $f(z) := \sum f_j z^j$ is of the form
$$f = (\psi_{11} - \varepsilon\psi_{21})^{-1}(\varepsilon\psi_{22} - \psi_{12}) = (\theta_{11}\varepsilon + \theta_{21})(\theta_{21}\varepsilon + \theta_{22})^{-1},$$
where $\varepsilon \in \mathbb{H}[[z]]$ is any power series subject to conditions (5) and such that $P_k^{\varepsilon} := I - \mathbf{T}_k^{\varepsilon}\mathbf{T}_k^{\varepsilon *} \succeq 0$ for all $k \geq 1$.

Function-theoretic setting

$\mathbb{H}[[z]]$ is a ring with operations

$$(f+g)(z)=\sum_{k=0}^{\infty} z^k(f_k+g_k) \quad \text{and} \quad (fg)(z)=\sum_{k=0}^{\infty} z^k\Big(\sum_{\ell=0}^{k} f_\ell g_{k-\ell}\Big).$$

Given $\rho>0$, let $\mathcal{H}_\rho$ be the ring of power series absolutely converging in the ball $\mathbb{B}_\rho=\{\alpha\in\mathbb{H}: \ |\alpha|<\rho\}$:

$$\mathcal{H}_\rho=\Big\{f(z)=\sum_{k=0}^{\infty} f_k z^k: \ \limsup_{k\to\infty}\sqrt[k]{|f_k|}\le 1/\rho\Big\}.$$

Any $f\in\mathcal{H}_\rho$ can be evaluated at any $\alpha\in\mathbb{B}_\rho$ on the left or on the right via (absolutely) converging series

$$f^{\boldsymbol{e}_\ell}(\alpha)=\sum_{k=0}^{\infty}\alpha^k f_k \quad \text{and} \quad f^{\boldsymbol{e}_r}(\alpha)=\sum_{k=0}^{\infty} f_k\alpha^k.$$

$\alpha \in \mathbb{H}$ is called a *left* or *right zero* of $f \in \mathcal{H}_\rho$ if respectively, $f^{\boldsymbol{e}_\ell}(\alpha) = 0$ or $f^{\boldsymbol{e}_r}(\alpha) = 0$. If $V \subset \mathbb{B}_\rho$ is a similarity class, then any $f \in \mathcal{H}_\rho$ either has no zeros in V or it has one left and one right zero in V, or $f^{\boldsymbol{e}_\ell}(\alpha) = f^{\boldsymbol{e}_r}(\alpha) = 0$ for all $\alpha \in V$.

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Therefore, for every $f \in \mathcal{H}_\rho$ and a similarity class $V \subset \mathbb{B}_\rho$, either $f^{\mathbf{e}_\ell}(\alpha) = c = f^{\mathbf{e}_r}(\alpha)$ for all $\alpha \in V$ or for any $\alpha \in V$ there is a unique $\alpha' \in V$ such that $f^{\mathbf{e}_r}(\alpha') = f^{\mathbf{e}_\ell}(\alpha)$.

Therefore, $f^{\mathbf{e}_\ell}(V) = f^{\mathbf{e}_r}(V)$ for any $f \in \mathcal{H}_\rho$ and $V \subset \mathbb{B}_\rho$.

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We now introduce the norm $\|f\|_\infty := \sup\limits_{\alpha \in \mathbb{B}_1} |f^{\mathbf{e}_\ell}(\alpha)| = \sup\limits_{\alpha \in \mathbb{B}_1} |f^{\mathbf{e}_r}(\alpha)|$ on $\mathcal{H}_1$ and define the *Schur class* $\mathcal{S}_{\mathbb{H}}$ to be

$$\mathcal{S}_{\mathbb{H}} := \{f \in \mathcal{H}_1 : \ \|f\|_\infty \leq 1\}.$$

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Alpay-B-Colombo-Sabadini (2015): A power series $f \in \mathbb{H}[[z]]$ belongs to $\mathcal{S}_{\mathbb{H}}$ if and only if $P_n = I_n - \mathbf{T}_n^f \mathbf{T}_n^{f*} \succeq 0$ for all $n \geq 1$.

Blaschke factors and products

Define the *Blaschke product* of degree n to be

$$f = \phi \cdot \mathbf{b}_{\alpha_1}\mathbf{b}_{\alpha_2}\cdots\mathbf{b}_{\alpha_n} \qquad (\alpha_i \in \mathbb{B}_1,\ |\phi| = 1)$$

where the *Blaschke factor* $\mathbf{b}_\alpha$ is the power series defined by

$$\mathbf{b}_\alpha(z) = (z-\alpha)(1-z\overline{\alpha})^{-1} = -\alpha+(1-|\alpha|^2)\sum_{k=0}^{\infty}\overline{\alpha}^k z^{k+1} \quad (\alpha \in \mathbb{B}_1).$$

Since $|\mathbf{b}_\alpha^{e_\ell}(\gamma)| = |\mathbf{b}_\alpha^{e_r}(\gamma)| \begin{array}{lll} <1 & \text{if} & |\gamma|<1, \\ =1 & \text{if} & |\gamma|=1, \end{array}$, Blaschke factors and products are in $\mathcal{S}_{\mathbb{H}}$.

Example: $\mathbf{b}_{\mathbf{i}/2}\mathbf{b}_{\mathbf{j}/2}\mathbf{b}_{\mathbf{k}/2}$

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B. (2021): A power series $f(z) = \sum f_k z^k \in \mathbb{H}[[z]]$ is a Blaschke product of degree k if and only if $P_n = I_n - \mathbf{T}_n^f\mathbf{T}_n^{f*}$ is positive semidefinite and $\operatorname{rank}(P_n) = \min(k, n)$ for all $n \geq 1$.

Carathéodory-Schur problem in $\mathcal{S}_{\mathbb{H}}$

Theorem: Given $f_0, \ldots, f_{n-1} \in \mathbb{H}$, there exists an

$$f(z) = f_0 + f_1 z + \ldots + f_{n-1} z^{n-1} + \ldots \in \mathcal{S}_{\mathbb{H}} \tag{6}$$

if and only if $P_n = I - \mathbf{T}_n^f \mathbf{T}_n^{f*} \succeq 0$. If $P_n \succ 0$, then the formula

$$f = (\theta_{11}\varepsilon + \theta_{21})(\theta_{21}\varepsilon + \theta_{22})^{-1} = (\psi_{11} - \varepsilon\psi_{21})^{-1}(\varepsilon\psi_{22} - \psi_{12}), \quad \varepsilon \in \mathcal{S}_{\mathbb{H}}$$

parametrizes all $f \in \mathcal{S}_{\mathbb{H}}$ subject to condition (6). Furthermore, f is a finite Blaschke product if and only if ε is. In this case, $\deg f = n + \deg \varepsilon$.

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- Linear fractional formulas make sense, since the bottom diagonal entry in P_n^{-1} is positive and $|\varepsilon_0| \leq 1$.
- If $\deg f = m$, then

$$\mathrm{rank}(I - \mathbf{T}_k^{\varepsilon} \mathbf{T}_k^{\varepsilon *}) = \mathrm{rank}\, P_{n+k} - n = \min\{m, n+k\} - n = \min\{m-n, k\}$$

Uniform approximation Carathéodory theorem

Theorem: Let $f \in \mathcal{S}_{\mathbb{H}}$. For any $\rho < 1$ and $\epsilon > 0$, there exists a finite Blaschke product B such that

$$|f^{\mathbf{e}_\ell}(\alpha) - B^{\mathbf{e}_\ell}(\alpha)| < \epsilon \quad \text{and} \quad |f^{\mathbf{e}_r}(\alpha) - B^{\mathbf{e}_r}(\alpha)| < \epsilon$$

for all $\alpha \in \overline{\mathbb{B}}_\rho$.

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Choose n such that $2\rho^n < \epsilon$ and assume that $f(z) = \sum_{j\geq 0} f_j z^j$ is not a finite Blaschke product so that $P_n = I - \mathbf{T}_n^f \mathbf{T}_n^{f*} \succ 0$. Then there is a finite Blaschke product B having the same first n coefficients as f. Then $g = \frac{1}{2}(f - B) \in \mathcal{S}_{\mathbb{H}}$, and $g(z) = z^n h(z)$ for some $h \in \mathcal{H}_1$. Since $\mathbf{T}_{n+m}^g = \begin{bmatrix} 0 & 0 \\ \mathbf{T}_m^h & 0 \end{bmatrix}$, for any $m \geq 1$, we have

$$I_{m+n} - \mathbf{T}_{n+m}^g \mathbf{T}_{n+m}^{g*} = \begin{bmatrix} I_n & 0 \\ 0 & I_m - \mathbf{T}_m^h \mathbf{T}_m^{h*} \end{bmatrix} \succeq 0 \quad \text{for all} \quad m \geq 1,$$

and hence, $h \in \mathcal{S}_{\mathbb{H}}$. Therefore, for any $\alpha \in \overline{\mathbb{B}}_\rho$,

$$|f^{\mathbf{e}_\ell}(\alpha) - B^{\mathbf{e}_\ell}(\alpha)| = 2|g^{\mathbf{e}_\ell}(\alpha)| = 2|\alpha|^n |h^{\mathbf{e}_\ell}(\alpha)| \leq 2\rho^n < \epsilon.$$

The generalized Schur class $\mathcal{S}_{\mathbb{H}}^{\kappa}$

Let us say that $f \in \mathbb{H}[[z]]$ belongs to the *generalized Schur class* $\mathcal{S}_{\mathbb{H}}^{\kappa}$ if the Hermitian matrices $P_n = I - \mathbf{T}_n^f \mathbf{T}_n^{f*}$ have κ negative eigenvalues counted with multiplicities:

$$\nu_-(I - \mathbf{T}_n^f \mathbf{T}_n^{f*}) = \kappa \quad \text{for all} \quad n \geq n_0.$$

The indefinite Carathéodory problem consists of finding

$$f(z) = f_0 + zf_1 + \ldots + f_{n-1}z^{n-1} + \ldots \in \mathcal{S}_{\mathbb{H}}^{\kappa} \tag{7}$$

with minimally possible κ (which is at least $\nu_-(P_n)$).

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with minimally possible κ (which is at least $\nu_-(P_n)$).If P_n is invertible, then $\kappa_{\min} = \nu_-(P_n)$. Moreover, the formula

$$f = (\theta_{11}\varepsilon + \theta_{21})(\theta_{21}\varepsilon + \theta_{22})^{-1} = (\psi_{11} - \varepsilon\psi_{21})^{-1}(\varepsilon\psi_{22} - \psi_{12})$$

with free parameter $\varepsilon \in \mathcal{S}_{\mathbb{H}}$ subject to conditions

$$\psi_{11,0} - \varepsilon_0\psi_{21,0} \neq 0 \iff \theta_{21,0}\varepsilon_0 + \theta_{22,0} \neq 0,$$

parametrizes all f of the form (7).

The singular case

Let us suppose that P_n is singular, $\mathrm{rank}(P_n) = d < n$, and let P_r ($r < n$) be the maximal invertible leading submatrix of P_n.

1. If $d = r$ (i.e., $\mathrm{rank}(P_n) = \mathrm{rank}(P_r)$), then there is a unique $f \in \mathcal{S}_{\mathbb{H}}^{\kappa}$ ($\kappa = \nu_-(P_n) = \nu_-(P_r)$) with initial coefficients $f_0, \ldots, f_{n-1}$.
2. If $d > r$, then the minimally possible κ equals

$$\kappa = \nu_-(P_n) + n - d = \nu_-(P_n) + \nu_0(P_n),$$

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   where ν_0 stands for the multiplicity of the zero eigenvalue.

• To get all $f \in \mathcal{S}_{\mathbb{H}}^{\kappa}$ with the first n coefficients equal to $f_0, \ldots, f_{n-1}$ in Case 2, we first choose arbitrary $f_n, \ldots, f_{2n-d}$ to reach the invertible matrix matrix P_{2n-d} and then apply the linear fractional formula to each such choice.

For any $f \in \mathcal{S}_{\mathbb{H}}^{\kappa}$, there is $n \geq \kappa$ such that P_n is invertible and $\nu_-(P_n) = \kappa$.

For such n, define the polynomial Θ as above and conclude that

$$f = (\theta_{11}\varepsilon + \theta_{21})(\theta_{21}\varepsilon + \theta_{22})^{-1}$$

for some $\varepsilon \in \mathcal{S}_{\mathbb{H}}$ such that $\theta_{21,0}\varepsilon_0 + \theta_{22,0} \neq 0$. Therefore, $\theta_{21}\varepsilon + \theta_{22}$ has no zeros in a neighborhood of the origin and therefore f converges absolutely in this neighborhood.

Thus, the power series $f \in \mathcal{S}_{\mathbb{H}}^{\kappa}$ can be left and right evaluated in a neighborhood of the origin giving rise to left and right regular functions $f^{\mathbf{e}_\ell}$ and $f^{\mathbf{e}_r}$.

Further elaboration may come from the Krein-Langer type factorization result: *For any* $f \in \mathcal{S}_{\mathbb{H}}^{\kappa}$ *there exist* $S_L, S_R \in \mathcal{S}_{\mathbb{H}}$ *and Blaschke products* B_L, B_R *of degree* κ *so that* f *admits coprime power-series factorizations*

$$f(z) = B_L(z)^{-1} S_L(z) = S_R(z) B_R(z)^{-1}.$$

Furthermore, $B_L B_L^{\sharp} = B_R B_R^{\sharp}$. If we denote by $\mathcal{Z}$ the zero set of the real Blaschke product $\widetilde{B} := B_L B_L^{\sharp} = B_R B_R^{\sharp}$, then the functions $f^{\mathbf{e}_\ell}$ and $f^{\mathbf{e}_r}$ admit meromorphic (semi-regular) extensions to $\mathbb{B} \backslash \mathcal{Z}$ by the formulas

$$f^{\mathbf{e}_\ell}(\alpha) = \widetilde{B}(\alpha)^{-1} (B_L^{\sharp} S_L)^{\mathbf{e}_\ell}(\alpha), \quad f^{\mathbf{e}_r}(\alpha) = (S_R B_R^{\sharp})^{\mathbf{e}_r}(\alpha) \widetilde{B}(\alpha)^{-1}.$$

Excluded parameters

Let us say that $\varepsilon \in \mathcal{S}_{\mathbb{H}}$ is an *excluded parameter* of order m of the linear fractional transformation

$$\mathbf{L}_{\Psi}[\varepsilon] := (\psi_{11} - \varepsilon\psi_{21})^{-1}(\varepsilon\psi_{22} - \psi_{12}), \quad \Psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix},$$

if $\boxed{\psi_{11} - \varepsilon\psi_{21} = z^m h}$ for some $h \in \mathbb{H}[[z]]$ with $h_0 \neq 0$.

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Theorem: There exists an excluded parameter $\varepsilon \in \mathcal{S}_{\mathbb{H}}$ of order m if and only if the $m \times m$ bottom principal submatrix of P_n^{-1} is either

(1) negative definite, in which case there are infinitely many excluded parameters of order m, or

(2) the maximal negative semidefinite bottom principal submatrix of P_n^{-1}, in which case there is a unique excluded parameter ε of order m.

Theorem: Let $\varepsilon \in \mathcal{S}_{\mathbb{H}}$ be an excluded parameter of order m. Then the power series $f_\varepsilon = \mathbf{L}_\Psi[\varepsilon]$ belongs to $\mathcal{S}_{\mathbb{H}}^{\kappa-m}$ and is of the form

$$f_\varepsilon(z) = f_0 + \ldots + f_{n-k-1}z^{n-k-1} + f_{\varepsilon,n-k}z^{n-k} + \ldots, \quad f_{\varepsilon,n-k} \neq f_{n-k}.$$

In other words, the $n-m$ first coefficients of f_ε are equal to prescribed $f_0, \ldots, f_{n-k-1}$, but $f_{\varepsilon,n-k}$ is different from the prescribed f_{n-k}.