Schur-Positivity for Generalized Nets

Ethan Shelburne

WM Groups, Analysis, Geometry Seminar

University of British Columbia

March 6th, 2024

Chromatic Polynomial (Birkhoff, 1912)

Definition

Given a graph G with vertex set $V(G)$, a proper coloring κ of G in k colors is

$$\kappa : V(G) \to \{1, 2, \ldots, k\}$$

such that

$$\kappa(v) \neq \kappa(u)$$

if there is an edge between u and v.

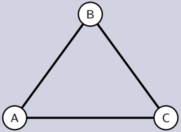
Example

Chromatic Polynomial (Birkhoff, 1912)

Definition

Given a graph G, the chromatic polynomial $\chi_G(k)$ is the number of proper colorings of G with k colors.

Example



has $\chi_G(k) = k(k-1)(k-2)$.

Chromatic Symmetric Function (Stanley, 1995)

Given a proper coloring κ of a graph with vertices $v_1, v_2, \ldots, v_N$ associate a monomial in commuting variables $x_1, x_2, x_3, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}.$$

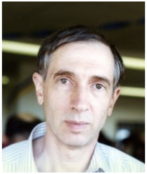
Example

gives $x_1 x_2$.

gives $x_2 x_1 = x_1 x_2$.

gives $x_1 x_3$.

Chromatic Symmetric Function (Stanley, 1995)



Definition

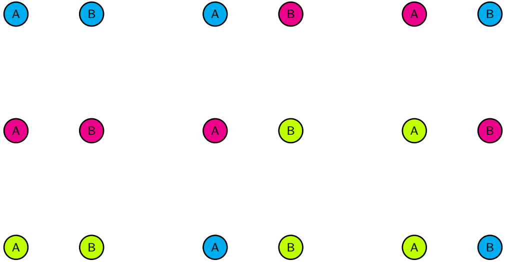
Given a graph G with vertices $v_1, v_2, \ldots, v_N$, the chromatic symmetric function is

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)},$$

where the sum ranges over all proper colorings κ of G.

Chromatic Symmetric Function (Stanley, 1995)

has $X_G = x_1^2 + x_2^2 + x_3^2 + \cdots + 2x_1x_2 + 2x_2x_3 + 2x_1x_3 + \cdots$.



The chromatic symmetric function is a generalization of the chromatic polynomial in the sense that

$$X_G(1^k, 0, \dots) = \chi_G(k)$$

where the lefthand side denotes $X_G(\mathrm{x})$ with the first k variables set as 1 and all other variables set as 0.

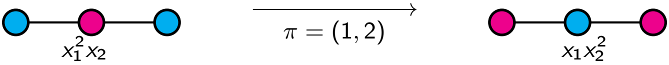
Symmetric Functions

Definition

A symmetric function is a formal power series f in countably many commuting variables $x_1, x_2, \ldots$ such that for all permutations π

$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

The chromatic symmetric function X_G is symmetric.



The algebra of symmetric functions is

$$\Lambda = \{f \in \mathbb{Q}[[x]] \;\mid\; f \text{ is symmetric}\}.$$

Partitions and Diagrams

Definition

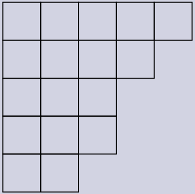
A partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ of N is a weakly decreasing list of positive integers whose sum is N. We then write $\lambda \vdash N$.

Definition

A Young diagram of $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ is an array of N boxes in left-justified rows such that row i contains λ_i boxes.

Example

The partition $(5, 4, 3, 3, 2) \vdash 17$ has the following Young diagram.



Semistandard Young Tableaux

Definition

A semistandard Young tableau (SSYT) of shape λ is a filling of a diagram with positive integers such that rows weakly increase (from left to right) and columns strictly increase (from top to bottom).

Example

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

We assign a weight to a given SSYT T which is the monomial

$$x^T = x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \cdots .$$

In our example,

$$x^T = x_1^3 x_2 x_4^2 x_5 x_6.$$

Schur Functions

Definition

The Schur function of partition λ is

$$s_\lambda = \sum_T x^T$$

where the sum spans over all SSYTs T of shape λ.

Example

$s_{21} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3 + \cdots$.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Schur-Positivity and e-Positivity

Definition

The elementary symmetric function e_i is given by

$$e_i = s_{(1^i)}.$$

Moreover, we define for a partition λ,

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}.$$

We have that

$$\{e_\lambda \mid \lambda \text{ is a partition}\} \qquad \text{and} \qquad \{s_\lambda \mid \lambda \text{ is a partition}\}$$

are both bases for Λ.

A graph G is Schur-positive if X_G is a nonnegative linear combination of s_λ. Likewise, G is e-positive if X_G is a nonnegative linear combination of e_λ.

$$\text{e-positivity of } G \implies \text{Schur-positivity of } G$$

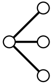
has

$X_G = e_{21} + 3e_3$ ✔

$X_G = 4s_{111} + s_{21}$ ✔

Three Major Conjectures: Definitions

The claw graph

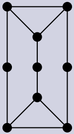
$K_{13} =$ 

is the smallest graph which is neither e-positive nor Schur-positive.

A graph G is claw-free if it has no copies of K_{13} as an induced subgraph.

Example

A claw-free graph.



Definition

An incomparability graph of a poset P is a graph $\mathrm{inc}(P)$ such that each element of P is assigned a vertex and u and v are adjacent if and only if u and v are incomparable in P.

Three Major Conjectures

1. (**The Stanley-Stembridge Conjecture**, 1993): All claw-free incomparability graphs are e-positive.

2. (**The Nonisomorphic Tree Conjecture**, 1995): No two nonisomorphic trees have the same chromatic symmetric function.

   Computationally confirmed on up to 29 vertices!

3. (**The Claw-Free Conjecture**, 1998): All claw-free graphs are Schur-positive.

Claw-Free Conjecture

Theorem (Gasharov, 1996)

If G is a claw-free incomparability graph, then G is Schur-positive.

Gasharov's proof method employs combinatorial object known as a P-array.

Definition

Let $P = (P, \prec)$ be a partially ordered set. A *P*-array is an array

$$\begin{array}{lll} a_{1,1} & a_{1,2} & \cdots \\ a_{2,1} & a_{2,2} & \cdots \\ \cdots & & \end{array}$$

of (distinct) elements in P, arranged in left-justified rows, and satisfying the following condition

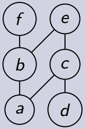
$$a_{i,j} \prec a_{i,j+1} \qquad \text{if} \qquad a_{i,j} \text{ and } a_{i,j+1} \text{ are defined.}$$

Note: A P-array is allowed to have empty rows. The shape of a P-array T is the sequence of row lengths given from top to bottom.

P-Arrays

Example

Consider the following poset P.



Below we depict several examples of P-arrays.

1	$a\ c$	$b\ f$		$a\ e$
2	d	d	$a\ b\ e$	
3			d	$b\ f$
4	$b\ f$	$a\ c$	f	c
5	e	e	c	d

P-Arrays

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we define $\pi(\lambda)$ to be the sequence

$$\lambda_{\pi(1)} - \pi(1) + 1, \ldots, \lambda_{\pi(\ell)} - \pi(\ell) + \ell.$$

Gasharov shows that if $G = \mathrm{inc}(P)$, Schur coefficients satisfy

$$[s_\lambda]X_G = \sum_{(\pi,T)\in A} \mathrm{sgn}(\pi)$$

where

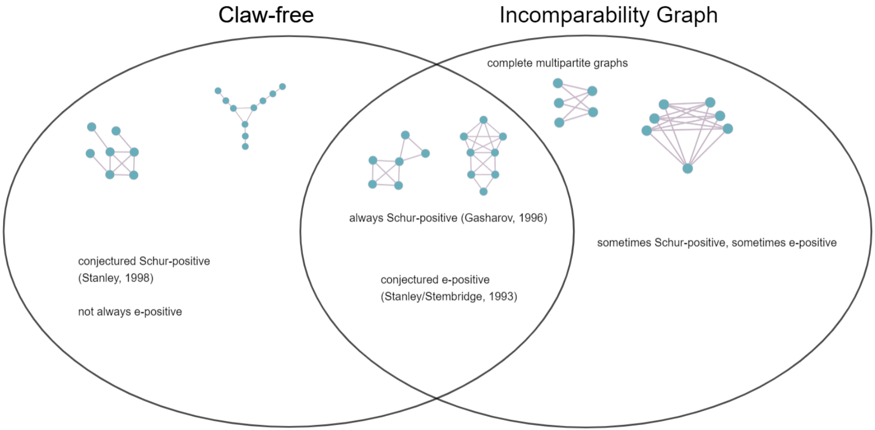
$$A = \{(\pi, T) \mid \pi \in S_\ell \text{ and } T \text{ is a } P\text{-array of shape } \pi(\lambda)\}.$$

For example,

1	$a\ c$	$b\ f$		$a\ e$
2	d	d	$a\ b\ e$	
3			d	$b\ f$
4	$b\ f$	$a\ c$	f	c
5	e	e	c	d

are of shape $\pi(2,1,1,1,1)$ for $\pi = (3,4)$, $\pi = (3,4)$, $\pi = (1,2)$, and $\pi = (2,3)$. Gasharov then constructs a sign-reversing involution on A to prove Schur-positivity for claw-free incomparability graphs!

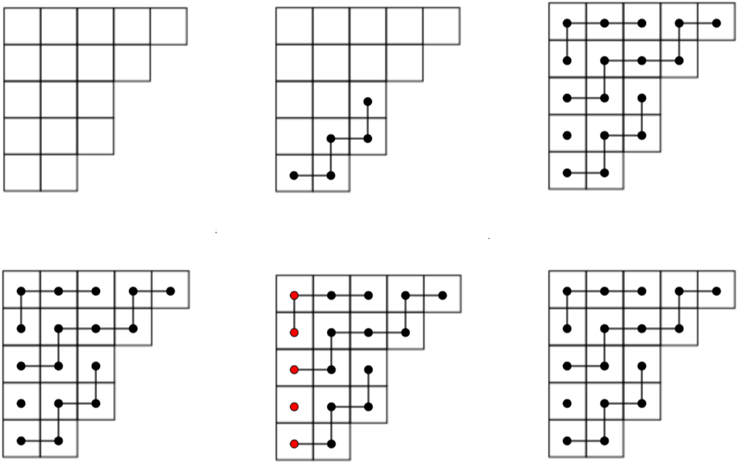
Research on Schur-Positivity of Chromatic Symmetric Functions



Question: What claw-free non-incomparability graphs can we show are Schur-positive?

Special Rim Hook Tabloids

Example: $\mathrm{sgn}(T) = (-1)^5 = -1.$



A rim hook is a sequence of connected cells in a Young diagram.

A New Combinatorial Object: SRH G-tabloids

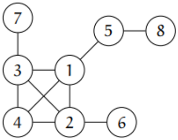
Definition

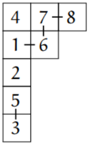
Let G be a graph with a partial order on the vertices such that nonadjacent vertices are comparable. We define an SRH G-tabloid to be an SRH tabloid such that the cells are filled with all the vertices of G and

- cells spanned by the same rim hook contain vertices which form a stable set,
- and, for each rim hook, reading the corresponding vertices in northeast order results in an increasing sequence.

Example: Equip the vertices of a non-incomparability graph with a total order.

G =





The sign of a SRH G-tabloid is the sign of the underlying SRH tabloid (so, in the above example, $\mathrm{sgn}(T) = 1$).

Combinatorial Interpretation for Schur-Coefficients

In (Wang-Wang, 2020), the authors introduce a combinatorial formula for all Schur coefficients of chromatic symmetric functions in terms of SRH tabloids.

We can reinterpret this result in terms of SRH G-tabloids.

Proposition (ES-SvW,2023)

Consider any graph G with a partial order on the vertices such that nonadjacent vertices are comparable. We have

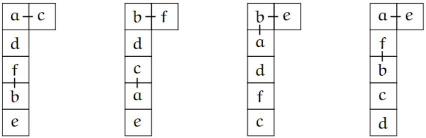
$$[s_\lambda]X_G = \sum_T \mathrm{sgn}(T)$$

where the sum ranges over all SRH G-tabloids of shape λ.

SRH G-Tabloids Generalize P-Arrays

In the case where $G = \mathrm{inc}(P)$, we found a sign-preserving bijection between P-arrays and SRH G-tabloids.

1	$a\ c$	$b\ f$		$a\ e$
2	d	d	$a\ b\ e$	
3			d	$b\ f$
4	$b\ f$	$a\ c$	f	c
5	e	e	c	d
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$



sgn(T)= -1 -1 -1 -1

We can now

- write Gasharov's proof in terms of SRH G-tabloids,
- obtain Wang-Wang's result as a corollary to Gasharov's proof.

Combinatorial Interpretation for Schur-Coefficients

Proposition (ES-SvW, 2023)

Consider any graph G on N ordered vertices and a partition $\lambda \vdash N$. We have

$$[s_\lambda]X_G = \sum_T \mathrm{sgn}(T)$$

where the sum ranges over all SRH G-tabloids of shape λ.

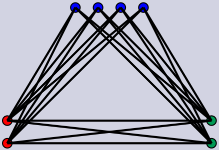
- In Wang-Wang's paper, they use this formula to prove certain graphs (squids, etc.) are not Schur-positive.
- Can we use this result to prove Schur-positivity?

Complete Multipartite Graphs

Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. A complete multipartite graph K_λ is a graph with k stable sets of vertices of respective sizes $\lambda_1, \ldots, \lambda_k$ with every possible edge between stable sets added.

Example

$K_{(4,2,2)} =$ 

In (Wang-Wang, 2020), the authors obtain a Schur-positivity classification for complete bipartite and tripartite graphs.

Theorem (ES-SvW, 2023)

Let K_λ be a complete multipartite graph with $\ell(\lambda) \geq 2$.
We have that K_λ is Schur-positive if and only if either $\lambda_i \in \{1, 2\}$ for $1 \leq i \leq k$, or $\lambda = (3, 2^j)$ for $j \geq 1$.

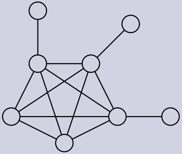
Generalized Nets

Recall: Our main goal is to make progress toward the Claw-Free Conjecture.

Definition

A generalized net $GN_{n,m}$, $n \geq 1$, $n \geq m \geq 0$, is a complete graph on n vertices with m degree one vertices appended to distinct vertices.

Example

$GN_{5,3} =$ 

- All generalized nets are claw-free.
- Generalized nets are not incomparability graphs for $m \geq 3$.
- Generalized nets $GN_{n,3}$ are never e-positive (Foley et al., 2018).

Theorem (ES-SvW, 2023): All generalized nets are Schur-positive.

Key Concepts of Proof

- Recall
$$[s_\lambda]X_G = \sum_T \operatorname{sgn}(T)$$
where the sum ranges over all SRH G-tabloids of shape λ.
- Choose a good partial order on the vertices (we use different orders at different points of the proof).
- Focus on coefficients for partitions $(\lambda_1, \ldots, 1^k)$, $k \geq 1$.
- Construct maps between SRH G-tabloids to find a recursive formula.

Proposition (ES-SvW, 2023)

Let λ be a partition and assume $\lambda_\ell = 1$. We then have

$$[s_\lambda]X_{GN_{n,m}} = m[s_{\lambda\setminus 1}]X_{GN_{n-1,m-1}\cup P_1} + (n-m)[s_{\lambda\setminus 1}]X_{GN_{n-1,m}} + m[s_{\lambda\setminus 1^2}]X_{GN_{n-1,m-1}}$$

for $n \geq 2, n \geq m \geq 1$. In this case, $\lambda \setminus 1^k$ denotes the partition λ with the last k 1s removed.

Note: We need entirely different arguments to cover coefficients $[s_\lambda]X_{GN_{n,m}}$ for which $\lambda_\ell \neq 1$.

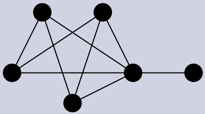
Generalized Nets with Edges Removed

Definition

The graph $GN^k_{n,m}$ is a generalized net $GN_{n,m}$ with k disjoint edges removed from (distinct) pairs of degree $n-1$ vertices.

Note: These graphs are not claw-free for $m \geq 1, k \geq 1$.

Example

$GN^2_{5,1} =$ 

Proposition (ES-SvW, 2023)

Let λ be a partition and assume $\lambda_\ell = 1$ and $n - 2k \geq m \geq 1$. We then have

$$[s_\lambda]X_{GN^k_{n,m}} = m[s_{\lambda\setminus 1}]X_{GN^k_{n-\mathbf{1},m-\mathbf{1}} \cup P_{\mathbf{1}}} + (n-m-2k)[s_{\lambda\setminus 1}]X_{GN^k_{n-\mathbf{1},m}}$$
$$+ 2k[s \setminus 1]X_{GN^{k-\mathbf{1}}_{n-\mathbf{1},m}} - k[s_{\lambda\setminus 1^2}]X_{GN^{k-\mathbf{1}}_{n-\mathbf{2},m}} + m[s_{\lambda\setminus 1^2}]X_{GN^k_{n-\mathbf{1},m-\mathbf{1}}}.$$

Generalized Nets with Edges Removed

Proposition (ES-SvW, 2023)

$$[s_{(2+h,2^{n-r-t-\mathbf{1}},1^t)}]X_{GN^r_{n,n-\mathbf{2}r-t+h}} = \binom{n-2r-t+h}{h} \cdot r!t!(n-2r-t)! \cdot \sum_{i=1}^{n-2r-t+1} (-1)^{i+1}\binom{r+i-1}{r}\binom{n-2r-i+1}{t}$$

Conjecture (ES-SvW)

$GN^k_{n,m}$ is Schur-positive for n sufficiently greater than m.

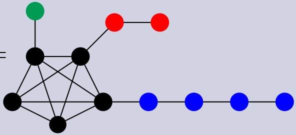
Generalized Spiders

Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. A generalized spider $GS_{n,\lambda}$ for $n \geq 1, n \geq k \geq 0$, is a complete graph K_n with paths of length $\lambda_1, \ldots, \lambda_k$ appended to distinct vertices in the complete graph.

Example

$GS_{5,(4,2,1)} =$



Proposition (ES-SvW, 2023)

Let λ be a partition and assume $\lambda_\ell = 1$ and $n \geq 3$, $m \geq 2$. We then have

$$[s_\lambda]X_{GN_{n,(\mathbf{2},\mathbf{1}^{m-1})}} = (m-1)[s_{\lambda\backslash 1}]X_{GN_{n-1,(\mathbf{2},\mathbf{1}^{m-1})}\cup P_1} + (n-m)[s_{\lambda\backslash 1}]X_{GN_{n-1,(\mathbf{2},\mathbf{1}^{m-1})}} + [s_{\lambda\backslash 1}]X_{GN_{n,(\mathbf{1}^{m-1})}\cup P_2} + [s_{\lambda\backslash 1^3}]X_{GN_{n-1,(\mathbf{1}^{m-1})}} + m[s_{\lambda\backslash 1^2}]X_{GN_{n-1,(\mathbf{2},\mathbf{1}^{m-2})}}.$$

Summary of Results and Future Directions

Significance of results?

- The SRH G-tabloid generalizes two combinatorial objects (P-array and SRH tabloid).
- Our proofs introduce new methods of proving Schur-positivity (since Gasharov's 1996 result, there have been very few results/methods toward the Claw-Free Conjecture).
- Generalized nets $GN_{n,3}$ are the first example we know of an infinite family of (non-tree) graphs which are proven to be Schur-positive but not e-positive.

What can we do next?

- Classify Schur-positivity for generalized nets with edges removed (and try removing other edges instead).
- Prove larger families of generalized spiders are Schur-positive.
- Prove line graphs (which are claw-free) are Schur-positive.

Final question: if the Claw-Free Conjecture is true, does there exist a proof employing a sign-reversing involution on SRH G-tabloids?

Thanks for listening!

References

[1] Angele M. Foley, Joshua Kazdan, Larissa Kroll, Sofia Martinez Alberga, Oleksii Melnyk, and Alexander Tenenbaum. Spiders and their kin: An investigation of Stanley's chromatic symmetric function for spiders and related graphs. 2018.

[2] V Gasharov. Incomparability graphs of (3 + 1)-free posets are s-positive. *Discrete Mathematics*, 157:107,125, 1996.

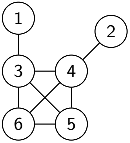
[3] Richard P Stanley and John R Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted position. *Journal of Combinatorial Theory, Series A*, 62:261,279, 1993.

[4] R.P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, 111(1):166,194, 1995.

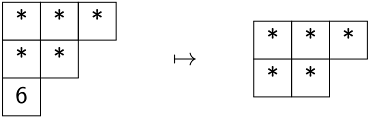
[5] David G. L. Wang and Monica M. Y. Wang. Non-Schur-positivity of chromatic symmetric functions. 2020.

Key Concepts of Proof

Call the degree one vertices pendants, the degree n vertices anchors, and the degree $n-1$ vertices buoys. We use a pendant-first labeling:



Assume $\lambda_\ell = 1$. If the vertex in the bottom cell is an anchor or buoy, it is in its own rim hook. We can remove it and obtain a smaller SRH G-tabloid.

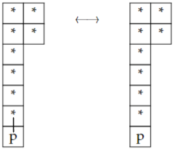


Hence, counting tabloids with anchors and buoys in the bottom position is equivalent to counting tabloids for smaller graphs and partitions. This gives us the following nonnegative terms in the recursion:

$$m[s_{\lambda\setminus\mathbf{1}}]X_{GN_{n-\mathbf{1},m-\mathbf{1}}\cup P_{\mathbf{1}}} \qquad \text{and} \qquad (n-m)[s_{\lambda\setminus\mathbf{1}}]X_{GN_{n-\mathbf{1},m}}$$

Key Concepts of Proof: Pendant in Bottom Position

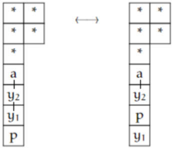
Case A: If an edge up from a pendant p is permissible:



sign reversing ✔

Case B: If an edge up from p is not permissible, there are two possibilities:

1. The vertex above p has a smaller label.
2. Otherwise, p may be below a hook which includes its anchor. Then we also have $p < y_1 < y_2 < \cdots < a$.



sign reversing ✔

the tabloids on the right

fall under subcase 1 ✔

Key Concepts of Proof

- We then focus on the subcase 1 tabloids which are not cancelled out be these maps.
- These all have at least two pendants in the bottom positions, which are in decreasing order (read from bottom to top).
- We thus apply similar maps iteratively and eventually obtain our recursive formula:

$$[s_\lambda]X_{GN_{n,m}} = m[s_{\lambda\setminus 1}]X_{GN_{n-\mathbf{1},m-\mathbf{1}}\cup P_{\mathbf{1}}} + (n-m)[s_{\lambda\setminus 1}]X_{GN_{n-\mathbf{1},m}} + m[s_{\lambda\setminus 1^2}]X_{GN_{n-\mathbf{1},m-\mathbf{1}}}.$$

- This process resembles a sorting algorithm for pendants in the bottom positions.

Note: We need entirely different arguments to cover coefficients s_λ for which $\lambda_\ell \neq 1$.